

Set Theory

Set:- A well defined collection of distinct objects is called a set.

The objects in a set can be numbers, people, letters, rivers, countries etc.

The objects in a set are called elements or members of the set.

Ex:-

1. The set of numbers 2, 4, 6, 8, 10

2. The set of letters a, e, i, o, u.

Note -

Sets are usually denoted by capital letters

A, B, C ... or X, Y, Z ... and the elements

are denoted by a, b, c, ... or x, y, z, ...

Standard notations for certain type of sets -

1. Natural Numbers $\mathbb{N} := \{1, 2, 3, \dots\}$

2. Integers $\mathbb{Z} := \{\dots -2, -1, 0, 1, 2, \dots\}$

3. Rational Numbers $\mathbb{Q} := \left\{ \frac{a}{b} \mid a, b \text{ are integers, } b \neq 0 \right\}$

4. Irrational Numbers $\mathbb{Q}^c$ $\mathbb{I} := \{ \text{a non rational number such as } \sqrt{2}, e, \pi, e^{\pi}, \dots \}$

5. Real Numbers $\mathbb{R} := \{ \text{a rational or an irr. no.} \}$

6. complex numbers $\mathbb{C} := \{ a + ib \mid a, b \in \mathbb{R}, i = \sqrt{-1} \}$

Also $\mathbb{Z}^+ \rightarrow$ Set of positive integers

$\mathbb{Q}^* \rightarrow$ Set of non-zero rational numbers

$2\mathbb{N} = E \rightarrow$ Set of even Integers

$D_n \rightarrow$ set of divisors of a positive Integer n .

(#) Cardinality or cardinal number of a set $\rightarrow$

The number of elements in a set A is called the cardinality or cardinal number of the set. It is denoted by $n(A)$ or $|A|$.

Equivalent Set:- A and B are said to be equivalent sets if $|A| = |B|$.

Ex:-

if $S = \{3, 5, 7, 9, 11\}$
Then $|S| = 5$.

Empty Set / Null set :-

A set having no element is called empty set. It is denoted by ϕ or $\{\}$.

eg $\rightarrow$ 1. The set of Integers b/w $\frac{9}{4}$ and $\frac{11}{4}$

2. The set of Integers less than 4 and greater than 8.

Finite and Infinite sets :-

A set is called finite if it has a distinct elements. otherwise set is called Infinite.

(#) Subset:- If every element of the set A is also an element of the set B then A is called subset of B . denoted by $A \subseteq B$ which is read as A is contained in B .
i.e. if $x \in A \Rightarrow x \in B$

Proper subset:- When $A \subseteq B$ and the set B has an element that is not in A . Then A is called proper subset of B and is denoted by $A \subset B$.

Equal Sets:- Two sets A and B are equal i.e. $A = B$ iff $A \subseteq B$ and $B \subseteq A$

Comparability:- Two sets A and B are said to be comparable if $A \subset B$ or $B \subset A$ or $A = B$.

(#) Power Set:- $\rightarrow$ Let X be a non-empty set. The set of all subsets of X is called the power set of the set X . It is denoted by $P(X)$.

Since $\phi \subseteq X$ and $X \subseteq X$, so $\phi, X \in P(X)$.
Particularly, $P(\phi) = \{\phi\}$.

Number of elements in Power Set:-

Let a set A has n elements. Then its Power Set $P(A)$ has 2^n elements

Proof:- For $0 \leq k \leq n$, there are nC_k ways to select a subset having k elements of the set A

Therefore, in Total, Power Set $P(A)$ contains

$$n C_0 + n C_1 + n C_2 + \dots + n C_{n-1} + n C_n = 2^n$$

Sets, as asserted.

Example:-1 Find the Power Set of the sets
(i) $\{a\}$, (ii) $\{a, b\}$ (iii) $\{a, b, c\}$

Soln:-

i.) let $A = \{a\}$

$$\text{So } P(A) = \{\phi, \{a\}\}$$

ii.) let $B = \{a, b\}$

$$\text{So } P(B) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$$

iii.) let $C = \{a, b, c\}$

$$\text{So } P(C) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$$

Example:-2 Find the Power Set of the sets

i.) $A = \{\phi, \{ \phi \}\}$

ii.) $B = \{a, \{a\}\}$

Soln:-

i.) Power set of A

$$P(A) = \{\phi, \{ \phi \}, \{ \{ \phi \} \}, \{ \phi, \{ \phi \} \}\}$$

ii.) Power set of B

$$P(B) = \{\phi, \{a\}, \{\{a\}\}, \{a, \{a\}\}\}$$

Imp

Ex:-3 If $A = \{\phi, b\}$ then find $A \cup P(A)$, $P(A)$ - Power set of A

Soln:- Since $P(A) = \{\phi, \{ \phi \}, \{b\}, A\}$

$$\therefore A \cup P(A) = \{\phi, \{ \phi \}, \{b\}, b, A\}$$

Operations on Set : →

1. Union of sets :- The union of two sets A and B is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
2. Intersection of sets :- The Intersection of two sets A and B is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
3. Difference of sets :- The difference of two sets A and B is defined by

$$A - B = \{x : x \in A, x \notin B\}$$
4. Complement of a set :- The complement of a set A is the set of elements which do not belong to A denoted by A^c .

$$A^c = U - A$$

where U is the universal set.
5. Symmetric difference of two sets :-
 Symmetric difference of two sets A and B is given by

$$A \oplus B = A \Delta B = (A - B) \cup (B - A)$$

or
$$A \Delta B = \{x \in X \mid x \in (A \cup B) - (A \cap B)\}$$

 where $X \neq \phi$, $A, B \in \mathcal{P}(X)$.

Note -

$$A \Delta B = B \Delta A \quad A \Delta A = \phi$$

$$A \Delta \phi = A, \quad A \Delta X = A', \quad A \Delta A' = X$$

Ques.

Let $U = \{1, 2, \dots, 8, 9\}$ be a universal set and the six sets given by

$$A = \{1, 2, 3, 4, 5\}, B = \{4, 5, 6, 7\},$$

$$C = \{5, 6, 7, 8, 9\}, D = \{1, 3, 5, 7, 9\}$$

$$E = \{2, 4, 6, 8\}, F = \{1, 5, 9\} \text{ Then}$$

Find the sets

(a.) $A \cup B, D \cup F$ (b.) B^c, D^c, E^c

(c.) $A - B, D - E, E - D$ (d.) $C \cup D, E \cup F, A \cup B$

(e.) $(A \cup C) - B, (B \cup C) - A$

Soln:-

(a.) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

$$D \cup F = \{1, 3, 5, 7, 9\}$$

(b.) $B^c = \{1, 2, 3, 8, 9\}, D^c = \{2, 4, 6, 8\}$

$$E^c = \{1, 3, 5, 7, 9\}$$

(c.) $A - B = \{1, 2, 3\}$

$$D - E = \{1, 3, 5, 7, 9\}$$

$$E - D = \{2, 4, 6, 8\}$$

(d.) $C \cup D = \{1, 3, 5, 6, 7, 8, 9\}$

$$E \cup F = \{1, 2, 4, 5, 6, 8, 9\}$$

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$

(e.) $(A \cup C) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

So $(A \cup C) - B = \{1, 2, 3, 8, 9\}$

$$B \cup C = \{4, 5, 6, 7, 8, 9\}$$

So $(B \cup C) - A = \{6, 7, 8, 9\}$

Laws of Algebra of Sets:- Let $A, B, C \in \mathcal{P}(X)$

1. Idempotent Laws : a.) $A \cup A = A$
b.) $A \cap A = A$
2. Associative Laws : a.) $(A \cup B) \cup C = A \cup (B \cup C)$
b.) $(A \cap B) \cap C = A \cap (B \cap C)$
3. commutative Laws : a.) $A \cup B = B \cup A$
b.) $A \cap B = B \cap A$
4. Distributive Laws : a.) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
b.) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. Identity Laws :
a.) $A \cup \phi = A$, $A \cup U = U$
b.) $A \cap \phi = \phi$, $A \cap U = A$
6. Involution law :
 $(A')' = A$
7. complement laws : a.) $A \cup A' = U$, $A \cap A' = \phi$
b.) $U' = \phi$, $\phi' = U$
8. De Morgan's law : a.) $(A \cap B)' = A' \cup B'$
b.) $(A \cup B)' = A' \cap B'$
9. a.) $A - (B \cup C) = (A - B) \cap (A - C)$
b.) $A - (B \cap C) = (A - B) \cup (A - C)$
10. Absorption law : a.) $A \cup (A \cap B) = A$
b.) $A \cap (A \cup B) = A$
11. $A \cap (B - C) = (A \cap B) - (A \cap C)$
12. a.) $A - B = A \cap B'$
b.) $B - A = B \cap A'$

Note:- a.) $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$
 $x \in A$ or $x \in B \Leftrightarrow x \in A \cup B$
 $x \notin A$ and $x \notin B \Leftrightarrow x \notin A \cup B$
 $x \notin A$ or $x \notin B \Leftrightarrow x \notin A \cap B$

Ques.1. Prove that if $A \subseteq B$ then $A \cup B = B$.

Soln:-

We know that $B \subseteq A \cup B$ — (1)

Now we have to show $A \cup B \subseteq B$

Let $x \in A \cup B \Rightarrow x \in A$ or $x \in B$
 $\Rightarrow x \in B$ as $A \subseteq B$

So $A \cup B \subseteq B$ — (2)

Hence (1) and (2) $\Rightarrow$

$$A \cup B = B$$

Ques.2. Prove that if $A \subseteq B$ then $A \cap B = A$.

Soln:-

We know that $A \cap B \subseteq A$ — (1)

Now we have to show $A \subseteq A \cap B$

Let $x \in A \Rightarrow x \in B$ as $A \subseteq B$
 $\Rightarrow x \in A \cap B$

i.e. $A \subseteq A \cap B$ — (2)

Hence from (1) and (2),

$$\boxed{A = A \cap B}$$

Ques.3 Prove that $A - B = A \cap B^c$.

Let $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$
 $\Leftrightarrow x \in A$ and $x \in B^c$
 $\Leftrightarrow x \in A \cap B^c$

Hence $A - B = A \cap B^c$.

Ques. 4. Let A, B, C be three non-empty sets.
Prove the following Identities:

$$1. (A - C) \cap (C - B) = \phi$$

$$2. A - (B \cap C) = (A - B) \cup (A - C)$$

$$3. (A^c \cup B^c)^c \cup (A^c \cup B)^c = A$$

Solⁿ -

① For. if $(A - C) \cap (C - B)$ is non-empty
let $x \in (A - C) \cap (C - B)$

$$\Rightarrow x \in (A - C) \text{ and } x \in (C - B)$$

However, the two conditions can not hold simultaneously because

$$x \in A - C \Rightarrow x \notin C$$

$$\text{and } x \in C - B \Rightarrow x \in C$$

which is a contradiction.

Therefore,

$$(A - C) \cap (C - B) = \phi$$

② we know that $X - Y = X \cap Y^c$

Therefore, we have

$$A - (B \cap C) = A \cap (B \cap C)^c$$

$$= A \cap (B^c \cup C^c) \quad (\text{by De Morgan law})$$

$$= (A \cap B^c) \cup (A \cap C^c)$$

(by distributive law)

$$= (A - B) \cup (A - C)$$

$$\textcircled{3} (A^c \cup B^c)^c \cup (A^c \cup B)^c = (A \cap B) \cup (A \cap B^c) \quad \begin{array}{l} \text{De Morgan's law} \\ \& \text{Involution law} \end{array}$$

$$= A \cap (B \cup B^c)$$

Distributive law

$$= A \cap U \quad \text{by complement law}$$

$$= \underline{\underline{A}}$$

#

Multiset : $\rightarrow$

Multisets are sets where an element can occur as a member more than once.

OR

A multiset (mset) is an unordered collection having multiple instances of the same object, where the multiplicity of an object may be zero, one, or more than one. We usually write a Multiset M as

$$M := [k_1 \cdot a_1, k_2 \cdot a_2, k_3 \cdot a_3, \dots]$$

where $k_i \in \mathbb{Z}^+$

where the multiplicity k_i of an object a_i in the multiset M is the number of times it appears in the collection.

The set $\{a_1, a_2, a_3, \dots\}$ is called the base set of the multiset M .

Example:- The two multisets A and B given by

$$A = [1, 1, 2, 2, 3, 4, 6, 6, 6]$$

$$B = [1, 2, 2, 2, 3, 3, 4, 4, 6]$$

can be written as

$$A = [2 \cdot 1, 2 \cdot 2, 3 \cdot 3, 4 \cdot 4, 3 \cdot 6]$$

and $B = [1 \cdot 1, 3 \cdot 2, 2 \cdot 3, 2 \cdot 4, 1 \cdot 6]$

Cardinality of multiset :-

The cardinality of a multiset is the sum of the multiplicities of all its elements.

#

Cartesian Product of sets :-

Ordered Pair :- an ordered pair, generally denoted by (a, b) , is a pair of elements a and b of two sets, which is ordered in sense that $(a, b) \neq (b, a)$ when $a \neq b$ and a and b are known as first and second coordinates in (a, b) .

Cartesian Product of two sets :-

Let A and B two non-empty sets. The Cartesian Product of A and B , denoted by $A \times B$, is the set of ordered pairs given by

$$A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$$

For Example -

$$\text{Let } A = \{a, b\} \text{ and } B = \{1, 2, 3\}$$

Then we have

$$A \times B = \{ (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3) \}$$

Note :- $A \times B \neq B \times A$

More generally if A and B are finite sets given by $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$

Then the set $A \times B$ contains mn elements

$$|A \times B| = |A| \times |B| = mn$$

The Product Set $A \times B$ in matrix form

	b_1	b_2	...	b_n
a_1	(a_1, b_1)	(a_1, b_2)	...	(a_1, b_n)
a_2	(a_2, b_1)	(a_2, b_2)	...	(a_2, b_n)
$\vdots$	$\vdots$	$\vdots$		
a_m	(a_m, b_1)	(a_m, b_2)	...	(a_m, b_n)

Properties of Cartesian Product

For the four sets A, B, C and D

1. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
2. $(A - B) \times C = (A \times C) - (B \times C)$
3. $(A \cup B) \times C = (A \times C) \cup (B \times C)$
4. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Ques. Let A, B, C be three non empty sets,
Prove that

1. $A \cap C = B \cap C$ and $A \cup C = B \cup C \Rightarrow A = B$
2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof:-

1.) we have,

$$\begin{aligned}
 A &= A \cap (A \cup C) && \text{by absorption} \\
 &= A \cap (B \cup C) && \text{by given condition} \\
 &= (A \cap B) \cup (A \cap C) && \text{by distributivity} \\
 &= (A \cap B) \cup (B \cap C) && \text{by given condition} \\
 &= (A \cap B) \cup (C \cap B) && \text{by commutativity} \\
 &= (A \cup C) \cap B && \text{by distributivity} \\
 &= (B \cup C) \cap B && \text{by given condition} \\
 &= (C \cup B) \cap B && \text{by commutativity} \\
 &= B && \text{by absorption}
 \end{aligned}$$

Hence

$$A \cup C = B \cup C \text{ and } A \cap C = B \cap C \Rightarrow A = B$$

$$\begin{aligned}
 2.) \quad A \times (B \cup C) &= \{(x, y) \mid x \in A \text{ and } (y \in B \cup C)\} \\
 &= \{(x, y) \mid x \in A \text{ and } (y \in B \text{ or } y \in C)\} \\
 &= \{(x, y) \mid (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)\} \\
 &= \{(x, y) \mid (x, y) \in A \times B \text{ or } (x, y) \in A \times C\} \\
 &= (A \times B) \cup (A \times C)
 \end{aligned}$$

$$\begin{aligned}
 3.) \quad A \times (B \cap C) &= \{(x, y) \mid x \in A \text{ and } y \in (B \cap C)\} \\
 &= \{(x, y) \mid x \in A \text{ and } (y \in B \text{ and } y \in C)\} \\
 &= \{(x, y) \mid (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\} \\
 &= \{(x, y) \mid (x, y) \in A \times B \text{ and } (x, y) \in A \times C\} \\
 &= (A \times B) \cap (A \times C)
 \end{aligned}$$

Ques. If $A = \{1, 4\}$, $B = \{4, 3\}$, $C = \{3, 6\}$ verify that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Soln. -

$$\begin{aligned}
 B \cup C &= \{4, 3, 6\} \\
 \text{So } A \times (B \cup C) &= \{1, 4\} \times \{4, 3, 6\} \\
 &= \{(1, 4), (1, 3), (1, 6), (4, 4), (4, 3), (4, 6)\}
 \end{aligned}$$

$$\begin{aligned}
 A \times B &= \{1, 4\} \times \{4, 3\} \\
 &= \{(1, 4), (1, 3), (4, 4), (4, 3)\}
 \end{aligned}$$

$$\begin{aligned}
 A \times C &= \{1, 4\} \times \{3, 6\} \\
 &= \{(1, 3), (1, 6), (4, 3), (4, 6)\}
 \end{aligned}$$

$$\begin{aligned}
 \text{So } (A \times B) \cup (A \times C) &= \{(1, 4), (1, 3), (1, 6), (4, 4), (4, 3), (4, 6)\}
 \end{aligned}$$

Hence

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Ques. If $A = \{1, 3, 5, 7\}$, $B = \{2, 3, 5, 8\}$

Then find $(A \cap B) \times (B - A)$

Soln. - $A \cap B = \{3, 5\}$, $B - A = \{2, 8\}$

$$\begin{aligned}
 \text{So } (A \cap B) \times (B - A) &= \{(3, 2), (5, 2), (3, 8), (5, 8)\}
 \end{aligned}$$

#

Relation or Binary Relation :-

Let A and B be non-empty sets. Then any subset R of the Cartesian product $A \times B$ is called a relation from A to B and is denoted by R .

$\therefore R$ is a relation from A to $B \Rightarrow R \subseteq A \times B$
and

$$R = \{(x, y) \mid x \in A, y \in B \text{ and } xRy\}$$

where

xRy denotes that x is related to y by R

$x \not R y$ denotes that x is not related to y

Ex:- Let $A = \{1, 2, 5\}$ and $B = \{2, 4\}$ be two given sets. Find out the Relation from A to B defined by "is less than".

Solⁿ:-

Given $A = \{1, 2, 5\}$

$B = \{2, 4\}$

$\therefore A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (5, 2), (5, 4)\}$

Then

$$R = \{(1, 2), (1, 4), (2, 4)\}$$

Domain of R :- The set of first coordinates of every element of R

$$\text{Domain } R = \{x \in A : (x, y) \in R \text{ for some } y \in B\}$$

Range of R :- The set of second coordinates of every element of R .

$$\text{Range } R = \{y \in B : (x, y) \in R \text{ for some } x \in A\}$$

Ex:- Let $A = \{2, 3, 5\}$ and $B = \{2, 4, 6, 10\}$
and $R = \{(2, 2), (2, 4), (2, 6), (2, 10), (3, 6), (5, 10)\}$

Then $\text{Dom } R = \{2, 3, 5\}$

$\text{Range } R = \{2, 4, 6, 10\}$

Total number of distinct Relation from A to B

$$= 2^{|A \times B|} = 2^{|A| \cdot |B|} = 2^{mn}$$

where $m = |A|$

$n = |B|$.

(#) Relation on a Set :-

A Relation R from a Set A to the set A itself is called a Relation on A .

i.e. $R \subseteq A \times A$

$$R = \{(x, y) : x, y \in A \text{ and } x R y\}$$

Ex:- Let $A = \{1, 2, 3\}$ find the Relation on A defined by $\leq$.

Soln:-

$$A \times A = \{1, 2, 3\} \times \{1, 2, 3\}$$

$$= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

Then Relation $\leq$ on A is given by

$$R = \overset{(1,1)}{\{(1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}}.$$

⑤ Identity Relation :-

The Identity Relation I_A on a Set A is defined by

$$I_A = \{(x, y) : x, y \in A \text{ and } x = y\}$$

$$= \{(x, x) : x \in A\}$$

Ex:-

$$\text{Let } A = \{a, b, c\}$$

$$I_A = \{(a, a), (b, b), (c, c)\}$$

⑥ Inverse Relation :-

Let R be any relation from a Set A to B . The Inverse of R , R^{-1} is the Relation from B to A defined by

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

Consequently

$$xRy = yR^{-1}x$$

Ex:- Let $A = \{2, 3, 5\}$ and $B = \{6, 8, 10\}$
and Relation R is defined by "x divides y"
Then

$$R = \{(2, 6), (2, 8), (2, 10), (3, 6), (5, 10)\}$$

and

$$R^{-1} = \{(6, 2), (8, 2), (10, 2), (6, 3), (10, 5)\}$$

Operations on Relation :-

1.) complement of a Relation:-

Let R be a Relation from set A to B .
Then the complement of Relation R denoted by $\bar{R}$ or R' is a Relation from A to B such that

$$\bar{R} = \{(a, b) : (a, b) \notin R\}$$

Ex:- Let R be Relation from A to B where

$$A = \{1, 2, 3\} \quad B = \{8, 9\}$$

$$\text{and } R = \{(1, 8), (2, 8), (1, 9), (3, 9)\}$$

Find the complement of R .

we have

$$A \times B = \{(1, 8), (1, 9), (2, 8), (2, 9), (3, 8), (3, 9)\}$$

$$\bar{R} = \{(2, 9), (3, 8), (3, 9)\}$$

2.) Intersection of Relation:-

The Intersection of two Relations R and S is defined by

$$R \cap S = \{(x, y) : xRy \text{ and } xSy\}$$

3. Union of Relation:-

The Union of two Relations R and S is defined by

$$R \cup S = \{(x, y) : xRy \text{ or } xSy\}$$

Ex:- Let $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (3, 4), (4, 3)\}$
& $R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$

Then

$$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (4,4), (3,4), (4,3), (1,2), (2,1)\}$$

$$R_1 \cap R_2 = \{(1,1), (2,2), (3,3), (4,4)\}$$

(#) Properties of Relations:-

1.) Reflexive Relation:-

A relation R on a set A is Reflexive if aRa for every $a \in A$
i.e. $(a,a) \in R \quad \forall a \in A$.

Ex:- if $R_1 = \{(1,1), (2,2), (2,3), (3,3)\}$ be a Relation on a set $A = \{1,2,3\}$ then R_1 is Reflexive since aRa for every $a \in A$.

2.) Irreflexive:-

A relation R on a set A is Irreflexive if $a \not R a$ for all $a \in A$
i.e. $(a,a) \notin R$ for all $a \in A$.

Ex:- The Relation $R_2 = \{(1,2), (1,3), (2,1), (2,3)\}$ on $A = \{1,2,3\}$ is Irreflexive.

3. Symmetric Relation:-

A relation R on a set A is Symmetric if $aRb \Rightarrow bRa$,
i.e. whenever $(a,b) \in R \Rightarrow (b,a) \in R \quad \forall a,b \in A$

Ex:- Relation $R_3 = \{(1,1), (1,2), (1,3), (2,2), (2,1), (3,1)\}$ on A is Symmetric

Asymmetric Relation:-

A Relation R on a Set A is asymmetric if $aRb \Rightarrow b \narrow R a$

i.e. whenever $(a, b) \in R \Rightarrow (b, a) \notin R$ for $a \neq b$.

- The Relation $R_1 = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$ on $A = \{1, 2, 3\}$ is an asymmetric Relation.

Antisymmetric Relation:-

A Relation R on a Set A is antisymmetric if aRb and $bRa \Rightarrow a = b \quad \forall a, b \in A$

i.e. $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b \quad \forall a, b \in A$

- The Relation $R = \{(1, 2), (2, 2), (2, 3)\}$ on Set $A = \{1, 2, 3\}$ is an Antisymmetric Relation.

Transitive Relation:-

A Relation R on a Set A is transitive if aRb and $bRc \Rightarrow aRc$

i.e. whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

① Equivalence Relation: -

A Relation R on a Set A is called an equivalence Relation or RST relation if it is reflexive, symmetric and Transitive i.e.

R is an equivalence relation on A if it has the following three properties:

1. $(a, a) \in R$ for all $a \in A$ (Reflexive)
2. $(a, b) \in R \Rightarrow (b, a) \in R$ (Symmetric)
3. (a, b) and $(b, c) \in R \Rightarrow (a, c) \in R$ (Transitive)

Ex:- If R be a Relation in the set of Integer $\mathbb{Z}$ defined by.
 $R = \{(x, y) : x \in \mathbb{Z}, y \in \mathbb{Z}, (x-y) \text{ is divisible by } 6\}$
 Then prove that R is an equivalence Relation.

Solⁿ:- i.) Reflexive - Let $x \in \mathbb{Z}$
 Then $x - x = 0$ which is divisible by 6
 Therefore $x R x \quad \forall x \in \mathbb{Z}$
 Hence R is Reflexive

ii.) Symmetric:- Let $x, y \in \mathbb{Z}$
 Then $x R y \Rightarrow x - y$ is divisible by 6
 $\Rightarrow -(x - y)$ is divisible by 6
 $\Rightarrow (y - x)$ is divisible by 6
 $\Rightarrow y R x$

Hence R is Symmetric.

iii) Transitive :- Let $x, y, z \in \mathbb{Z}$

Then xRy and $yRz \Rightarrow x$ is divisible by y
and y is divisible by z

$\Rightarrow x - y$ is divisible by 6 and
 $y - z$ is divisible by 6

$\Rightarrow [(x - y) + (y - z)]$ is divisible by 6

$\Rightarrow x - z$ is divisible by 6

$\Rightarrow xRz$

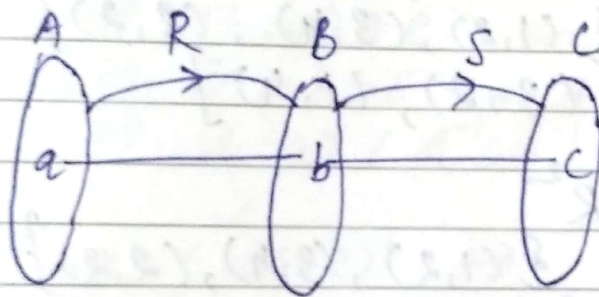
Hence, R is Transitive

Thus R is an equivalence Relation.

(#) Composite Relation: -

Let A, B and C be three non-empty sets.
 Suppose R be a relation from A to B
 and S be a relation from B to C
 Then the composite relation of the R and S
 is a relation from A to C and is
 denoted by $S \circ R$, and defined as

$$S \circ R = \{ (a, c) : \exists \text{ an element } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S \}$$



If $(a, b) \in R$
 $(b, c) \in S$ Then $(a, c) \in S \circ R$

Note - $R \circ R = R^2$, $R^2 \circ R = R^3$

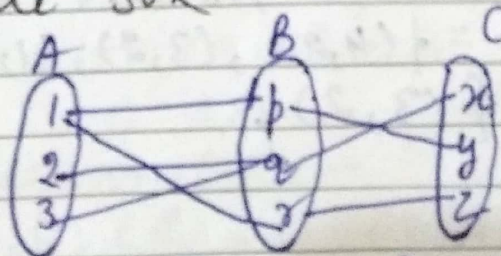
Ex! - Let $A = \{1, 2, 3\}$, $B = \{p, q, r\}$, $C = \{x, y, z\}$

and let

$$R = \{ (1, p), (1, r), (2, q), (3, q) \}$$

and $S = \{ (p, y), (q, x), (r, z) \}$ then
 compute $S \circ R$.

Solⁿ: -



$$S \circ R = \{ (1, y), (1, z), (2, x), (3, x) \}$$

Ex:- Let $R = \{(1, 2), (3, 4), (2, 2)\}$
and $S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$
Find $S \circ R$, $R \circ S$, $R \circ R$, $R \circ (S \circ R)$, $(R \circ S) \circ R$

Solⁿ:-

i.) $S \circ R$:-

$$R = \{(1, 2), (3, 4), (2, 2)\}$$

$$S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$$

$$\therefore S \circ R = \{(1, 5), (3, 2), (2, 5)\}$$

ii.) $R \circ S$:-

$$S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$$

$$R = \{(1, 2), (3, 4), (2, 2)\}$$

$$\therefore R \circ S = \{(4, 2), (3, 2), (1, 4)\}$$

iii.) $R \circ R = R^2$

$$R = \{(1, 2), (3, 4), (2, 2)\}$$

$$R = \{(1, 2), (3, 4), (2, 2)\}$$

$$R \circ R = R^2 = \{(1, 2), (2, 2)\}$$

iv.) $R \circ (S \circ R) \rightarrow$

$$S \circ R = \{(1, 5), (3, 2), (2, 5)\}$$

$$R = \{(1, 2), (3, 4), (2, 2)\}$$

$$R \circ (S \circ R) = \{(3, 2)\}$$

v.) $(R \circ S) \circ R \rightarrow$

$$R = \{(1, 2), (3, 4), (2, 2)\}$$

$$R \circ S = \{(4, 2), (3, 2), (1, 4)\}$$

$$(R \circ S) \circ R = \{(3, 2)\}$$

(H) Reversal law in composite relation →

Theorem:- Let R be a relation from the set A to the set B and S be a relation from the set B to set C , Then

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$$

Proof:-

$$\text{Let } (c, a) \in (S \circ R)^{-1}$$

$$\Leftrightarrow (a, c) \in (S \circ R) \quad \forall a \in A, c \in C$$

$$\Leftrightarrow \exists \text{ an element } b \text{ s.t.}$$

$$(a, b) \in R \text{ and } (b, c) \in S$$

$$\Leftrightarrow (b, a) \in R^{-1} \text{ and } (c, b) \in S^{-1}$$

$$\Leftrightarrow (c, b) \in S^{-1} \text{ and } (b, a) \in R^{-1}$$

$$\Leftrightarrow (c, a) \in R^{-1} \circ S^{-1}$$

$$\therefore (c, a) \in (S \circ R)^{-1} \Leftrightarrow (c, a) \in R^{-1} \circ S^{-1}$$

$$\text{Thus } (S \circ R)^{-1} = R^{-1} \circ S^{-1}$$

(H) Order of Relation →

- 1.) Partial order Relation
- 2.) Total order Relation.

(1) Partial order Relation →

A relation R on a set A is called partial order relation if R is

- 1.) Reflexive :- $(a, a) \in R, \forall a \in A$
i.e. $aRa \quad \forall a \in A$

- 2.) Antisymmetric :-

$$(a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b, \forall a, b \in A$$

$$\text{i.e. } aRb \text{ and } bRa \Rightarrow a = b$$

3.) Transitive:- $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$
 i.e. aRb & $bRc \Rightarrow aRc$

(#) POSET or Partial ordered Set

The set A together with a partially order relation R on the set A
 i.e. (A, R) is called Partial order set or POSET.

Ex:- Show that the Relation
 $R = \{(x, y) : x \geq y\}$ where $x, y \in I^+$
 the set of positive Integers, is a partial order Relation.

Soln:-

Let $x, y, z \in I^+$
 and given $R = \{(x, y) : x \geq y\}$.

i.) Reflexive:- Let $x \in I^+$
 Since $x \geq x \Rightarrow (x, x) \in R, \forall x \in I^+$
 $\Rightarrow xRx, \forall x \in I^+$
 $\Rightarrow R$ is Reflexive

ii.) Anti-Symmetric:- Let $x, y \in I^+$
 Now $(x, y) \in R$ and $(y, x) \in R$
 $\Rightarrow x \geq y$ and $y \geq x$
 $\Rightarrow x = y$
 $\Rightarrow (x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$
 $\therefore R$ is antisymmetric

3.) Transitive:- Let $x, y, z \in I^+$
 Let $(x, y) \in R$ and $(y, z) \in R$
 $\Rightarrow x \geq y$ and $y \geq z$
 $\Rightarrow x \geq z$
 $\Rightarrow (x, z) \in R$
 $\therefore (x, y) \in R \text{ \& } (y, z) \in R$
 $\Rightarrow (x, z) \in R$.

$\therefore R$ is Transitive
 Hence R is a Partial order Relation on I^+
 Thus (I^+, R) is a POSET.

④ Total order Relation:-

Consider the Relation R on the Set A .

If $\forall a, b \in A$, we have
 either $(a, b) \in R$
 or $(b, a) \in R$
 or $a = b$

Then R is called total order relation on A .

Ex:- Show that the Relation ' $<$ ' (less than) defined on N , the set of +ve Integers is neither an equivalence relation nor partially ordered relation but is a Total order relation.

Soln:-

Let $a, b \in N$
 $R = \{(a, b) : a < b\}$

1.) Reflexive:- Let $a \in N$
 Since $a \not< a$
 $\Rightarrow (a, a) \notin R \Rightarrow a \not R a$
 $\Rightarrow R$ is not reflexive

∴ The Relation ' $<$ ' is not Reflexive
 & Hence it is neither an equivalence
 relation nor the partial order relation.
 But

as $\forall a, b \in \mathbb{N}$
 we have

either $a < b$ $(a, b) \in R$
 or $b < a$ $(b, a) \in R$
 or $a = b$

So, the Relation is a total ordered
 relation.

⑧ composition of relations SOR by using
 matrix of Relations. $\rightarrow$

Matrix Representation of Relation :-

Let R be the Relation from set A to set B
 where

$$A = \{a_1, a_2, \dots, a_m\}$$

$$B = \{b_1, b_2, \dots, b_n\}$$

be finite sets having m and n elements resp.
 Then R can be represented by mn matrix
 and defined as

$$M_R = [m_{ij}]_{m \times n}$$

$$= \begin{cases} 1, & \text{if } aRb \text{ i.e. } (a, b) \in R \\ 0, & \text{if } aRb \text{ i.e. } (a, b) \notin R \end{cases}$$

The matrix M_R is called matrix of
 Relation R .

Ex:-1 Let R be the Relation from set $A = \{1, 3, 4\}$ to itself and defined by

$$R = \{(1, 1), (1, 3), (3, 3), (4, 4)\}$$

then find Relation matrix.

Sol:-

$$M_R = \begin{matrix} & \begin{matrix} 1 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Ex:-2.

Let $A = \{1, 2, 3, 4, 8\}$, $B = \{1, 4, 6, 9\}$

Let $a R b$ iff $a|b$ i.e. a divides b
find the Relation matrix M_R .

Sol:-

Here $R = \{(1, 1), (1, 4), (1, 6), (1, 9), (2, 4), (2, 6), (3, 6), (3, 9), (4, 4)\}$

$$M_R = \begin{matrix} & \begin{matrix} 1 & 4 & 6 & 9 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 8 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Ex:- Let $P = \{2, 3, 4, 5\}$. consider the Relation R and S defined by

$$R = \{(2, 2), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (5, 3)\}$$

$$S = \{(2, 3), (2, 5), (3, 4), (3, 5), (4, 2), (4, 5), (5, 2), (5, 5)\}$$

Find the matrices of the above relations
Use matrix to find the composition relation of R and S .

Solⁿ:

The matrices of the relation R and S are

$$M_R = \begin{matrix} & \begin{matrix} 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

and

$$M_S = \begin{matrix} & \begin{matrix} 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

The composite Relation of R and S is SOR

So Multiplying the matrix M_S with M_R to obtain the matrix $M_S \times M_R$

$$M_S \times M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{matrix} & \begin{matrix} 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix} \end{matrix}$$

The non-zero entries in the matrix $M_S \times M_R$ tell the elements related in composition relation SOR of R and S.

Hence

$$SOR = \{ (2,3), (2,4), (2,5), (3,3), (3,5), (4,2), (4,3), (4,4), (4,5), (5,2), (5,3), (5,4), (5,5) \}$$

(#) Total no. of Relations :-

Let A be a set with n elements i.e. $|A| = n$
Then

- 1.) Total no. of Relations = 2^{n^2}
- 2.) Total no. of Reflexive relation = $2^{n(n-1)}$ or 2^{n^2-n}
- 3.) Total no. of Irreflexive relation = 2^{n^2-n}
- 4.) Total no. of Symmetric relation = $2^{n \cdot \frac{(n+1)}{2}}$
- 5.) Total no. of Asymmetric relation = $3^{\frac{n(n-1)}{2}}$
- 6.) Total no. of Antisymmetric relation = $2^n \cdot 3^{\frac{n(n-1)}{2}}$
- 7.) Total no. of ^{both} Reflexive and Symmetric = $2^{\frac{1}{2}(n^2-n)}$

(#) Dirgraph / Directed graph of Relation $R \rightarrow$

If A is finite set and R is a Relation on A , then we can also represent R pictorially as

1. Draw a small circle for each element of A and label the circle with corresponding elements of A . These circles are called vertices or nodes.
2. Draw an arrow called an edge, from vertex a_i to $a_j \Leftrightarrow a_i R a_j$.

The Resulting pictorial Representation of R is called a Directed graph or Digraph of R .

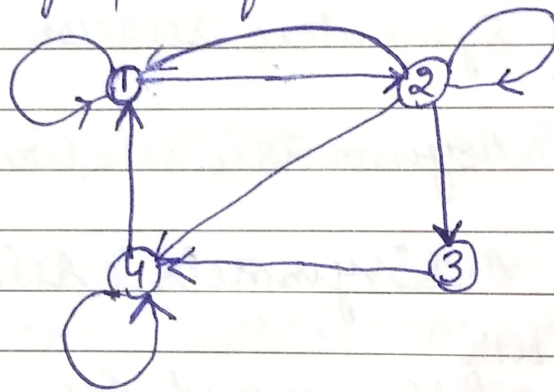
Ex:- Let $A = \{1, 2, 3, 4\}$

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1), (4, 4)\}$

Construct the digraph of R .

Soln:-

The digraph of R is



Closure of Relations : $\rightarrow$

Let R be a Relation on a Set A .

R may or may not have some Property P , such as reflexivity, symmetry or transitivity.

If we add some pairs then we have the desired Property. The Smallest relation S on A that contains R and possess the desired Property P is called closure of R with respect to that Property.

1. Reflexive closure : —

Let R be a Relation on a Set A .

The Reflexive closure $R^{(r)}$ of a Relation R is the smallest reflexive relation that contains R as a Subset. and is given by

$$R^{(r)} = R \cup \Delta_A$$

where $\Delta_A = \{(a, a) : a \in A\}$ is the diagonal Relation on A .

Exo -

Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$

Then find the Reflexive closure of R .

Sol : —

Reflexive closure of R

$$R^{(r)} = R \cup \Delta_A$$

$$\Delta_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$\begin{aligned} \text{So } R^{(r)} &= \{(1, 1), (2, 2), (1, 2), (2, 1)\} \cup \{(1, 1), (2, 2), (3, 3), (4, 4)\} \\ &= \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\} \end{aligned}$$

2. Symmetric closure :-

Let R be a relation on a set A , which is not symmetric. The symmetric closure $R^{(s)}$ is the smallest symmetric relation that contains R as a subset. and is given by

$$R^{(s)} = R \cup R^{-1}$$

where $R^{-1} = \{(y, x) : (x, y) \in R\}$
be the Inverse of R .

Ex:-

If $R = \{(1, 2), (2, 1), (2, 2), (3, 1), (4, 3)\}$
be a Relation on $S = \{1, 2, 3, 4\}$. Find
the symmetric closure of R .

Sol:-

The symmetric closure of R can be found by taking the union of R and R^{-1} .

Now $R^{-1} = \{(2, 1), (1, 2), (2, 2), (1, 3), (3, 4)\}$

$$\begin{aligned} \text{So } R^{(s)} &= R \cup R^{-1} \\ &= \{(1, 2), (2, 1), (2, 2), (3, 1), (4, 3)\} \cup \\ &\quad \{(2, 1), (1, 2), (2, 2), (1, 3), (3, 4)\} \\ &= \{(1, 2), (2, 1), (2, 2), (3, 1), (4, 3), (1, 3), (3, 4)\} \end{aligned}$$

3. Transitive closure :-

The Relation obtained by adding the least number of ordered pairs to ensure Transitivity is called the transitive closure of the Relatⁿ R . and is given by

$$R^* = R \cup R^2 \cup \dots \cup R^n$$

where $n = |A|$ & R be a Relatⁿ on A .

Ex:- Let $A = \{1, 2, 3, 4\}$
and $R = \{(1, 2), (2, 3), (3, 4)\}$
be a Relation on A . Find transitive closure R^*
using composition of R .

Solⁿ: given $R = \{(1, 2), (2, 3), (3, 4)\}$

→ New $R^2 = R \circ R$

$$R = \{(1, 2), (2, 3), (3, 4)\}$$

$$R = \{(1, 2), (2, 3), (3, 4)\}$$

So $R^2 = \{(1, 3), (2, 4)\}$

→ $R^3 = R^2 \circ R$

$$R = \{(1, 2), (2, 3), (3, 4)\}$$

$$R^2 = \{(1, 3), (2, 4)\}$$

So $R^3 = \{(1, 4)\}$

→ $R^4 = R^3 \circ R$

$$R = \{(1, 2), (2, 3), (3, 4)\}$$

$$R^3 = \{(1, 4)\}$$

So $R^4 = \emptyset$

Hence the transitive closure of R

$$R^* = R \cup R^2 \cup R^3 \cup R^4$$

$$= \{(1, 2), (2, 3), (3, 4)\} \cup \{(1, 3), (2, 4)\} \cup \{(1, 4)\} \cup \emptyset$$

$$= \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\}$$

(#) Warshall's Algorithm :-

An efficient method for computing the Transitive closure of a Relation is known as Warshall's algorithm named after Stephen Warshall.

Step 1:- Let n be the number of elements in a given set A .
To find transitive closure of Relation R on A , maximum n warshall sets can be find
 $W_0, W_1, \dots, W_n$ where $W_0 = M_R$
and W_n is the required matrix to find transitive closure.

Step 2:- Procedure to compute W_k from W_{k-1} .

1. copy 1 entries from W_{k-1} in W_k with same position.
2. In k^{th} row and k^{th} column of W_{k-1} ,
See k^{th} column and check the position of 1
and put below p_i .
See k^{th} row and check the position of 1
and put below q_i .
3. mark entries in W_k as 1 for (p_i, q_i)
of W_k if there are not already 1.

Step 3:- stop when W_n is obtained and it
is the required transitive closure

Ex:-1. Let $A = \{4, 6, 8, 10\}$
 $R = \{(4, 4), (4, 10), (6, 6), (6, 8), (8, 10)\}$
 is a Relation on Set A. Determine the transitive closure of R using warshall's algorithm

Soln:- Given $A = \{4, 6, 8, 10\}$
 and $R = \{(4, 4), (4, 10), (6, 6), (6, 8), (8, 10)\}$

$$M_R = \begin{matrix} & \begin{matrix} 4 & 6 & 8 & 10 \end{matrix} \\ \begin{matrix} 4 \\ 6 \\ 8 \\ 10 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Now we have $|A| = n = 4$

Thus we have to find warshall's sets W_0, W_1, W_2, W_3, W_4 .

Here

$$W_0 = M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. To compute W_1 - Transfer all 1's from W_0 to W_1 ,
 The locations of non-zero entries in C_i :- $P_i = 1$
 The locations of non-zero entries in R_i :- $Q_i = 1, 4$

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1, 1), (1, 4)$$

2. To compute W_2 - Transfer all 1's from w_1 to w_2

Positions of the non zero entries in C_2 : $p_i = 2$

Positions of non zero entries in R_2 : $q_i = 2, 3$

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2, 2), (2, 3)$$

3. To compute W_3 ! - Transfer all 1's from w_2 to w_3

Positions of non zero entries in C_3 : $p_i = 2$

Positions of non zero entries in R_3 : $q_i = ?$

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2, 4)$$

4. To compute W_4 ! - Transfer all 1's from w_3 to w_4

Positions of non-zero entries in C_4 : $p_i = 1, 2, 3$

Positions of non-zero entries in R_4 : $q_i =$ —
No New Pairs forms

So $W_4 = W_3$

$$= \begin{matrix} & 4 & 6 & 8 & 10 \\ \begin{matrix} 4 \\ 6 \\ 8 \\ 10 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$\therefore R^* = \{ (4, 4), (4, 10), (6, 6), (6, 8), (6, 10), (8, 10) \}$$

Ex:-2. Let $A = \{1, 2, 3, 4\}$ & $R = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on A . Find transitive closure R^* by warshall's algorithm.

Soln:-

Here $|A| = 4$

So we have to find warshall's sets W_0, W_1, W_2, W_3, W_4 .

$$R = \{(1, 2), (2, 3), (3, 4)\}$$

$$\therefore M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}_{4 \times 4}$$

$$\text{Now } W_0 = M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1.) $W_1 \rightarrow$

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} C_1 & R_1 \\ p_i & q_i \\ \emptyset & 2 \end{matrix}$$

No new pairs formed

2.) $W_2 \rightarrow$

$$W_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} C_2 & R_2 \\ p_i & q_i \\ 1 & 3 \\ (1, 3) \end{matrix}$$

3.) $W_3 \rightarrow$

$$W_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C_3	R_3
P_i	Q_i
1	4
2	

$(1, 4), (2, 4)$

4.) $W_4 \rightarrow$

$$W_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C_4	R_4
P_i	Q_i
1	ϕ
2	
3	

No new pairs formed

Hence the Required Transitive closure is

$$R^* = \{ (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) \}$$

Equivalence classes $\rightarrow$

gmp

Suppose R is an equivalence relation on a set A . For each x in A , the equivalence class of x denoted by $[x]$ is the set of all elements y in A such that x is related to y by R .

$$\text{i.e. } [x] = \{y \in A : (x, y) \in R\}$$

Ex:- Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3), (4, 4)\}$ is equivalence relation. Determine equivalence classes.

Sol:- R is an equivalence relation on A

$$[1] = \{1, 2\}$$

$$[2] = \{1, 2\}$$

$$[3] = \{3\}$$

$$[4] = \{4\}$$

Quotient Set $\rightarrow$

The collection of all equivalence classes of elements of set A under an equivalence relation R is called Quotient Set, denoted by A/R

$$\text{i.e. } A/R = \{[x] : x \in A\}$$

Ex:- In above example,

$$A/R = \{[1], [2], [3], [4]\}$$

$$\text{or } A/R = \{\{1, 2\}, \{3\}, \{4\}\}$$

Partitions of a Set $\rightarrow$

A partition of a set A is a set of non-empty subsets of A denoted by $\{A_1, A_2, \dots, A_k\}$ such that

$$i.) A_1 \cup A_2 \cup \dots \cup A_k = A$$

$$ii.) A_i \cap A_j = \emptyset \text{ for } i \neq j$$

Ex:- Let $A = \{1, 2, 3, 4\}$

$R = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3), (4, 4)\}$
Then

$$[1] = \{1, 2\}, [2] = \{1, 2\}$$

$$[3] = \{3\}, [4] = \{4\}$$

$$\text{So } P_1 = \{1, 2\}$$

$$P_2 = \{3\}$$

$$P_4 = \{4\}$$

$$P_1 \cup P_2 \cup P_3 = A$$

$$\& P_i \cap P_j = \emptyset \text{ for } i \neq j.$$

POSETS & Lattices

Date: / /

Page: -

Partially ordered Sets (POSETS) :- $\rightarrow$

A Relation R on a Set S is called a Partial order if it is Reflexive, antisymmetric and transitive i.e. for $a, b, c \in S$

- 1.) $aRa, \forall a \in S$ (Reflexive)
- 2.) aRb and $bRa \Rightarrow a=b$ (Antisymmetric)
- 3.) aRb and $bRc \Rightarrow aRc$ (Transitive)

Then "A set S together with a partial order Relation R is called a Partially ordered Set or Poset". It is denoted by (S, R) .

* The Relation R is often denoted by $\leq$ (different from the usual less than or equal to $\leq$)
Hence a poset is denoted by $(S, \leq)$

* In a poset $(S, \leq)$
 $a \leq b$ means 'a precedes b' and 'b succeeds a'.
 $a < b$ means 'a strictly precedes b' and 'b strictly succeeds a'.

Important Posets: -

- 1.) The set of Integers $\mathbb{Z}$ with relation $\geq$ i.e. $(\mathbb{Z}, \geq)$ is a Poset.

Since i.) $a \geq a, \forall a \in \mathbb{Z}$ (Reflexive)
ii.) $a \geq b$ and $b \geq a \Rightarrow a=b$ (Antisymmetric)
iii.) $a \geq b$ and $b \geq c \Rightarrow a \geq c$ (Transitive)
Hence $(\mathbb{Z}, \geq)$ is a Poset.

2.) The Power Set $P(S)$ of a set S together with the inclusion Relation $\subseteq$ i.e. $(P(S), \subseteq)$ is a Poset.

Since for A, B and $C \in P(S)$

i.) $A \subseteq A$, $\forall A \in P(S)$

$\Rightarrow \subseteq$ is reflexive

ii.) $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

$\Rightarrow \subseteq$ is Antisymmetric

iii.) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

$\Rightarrow \subseteq$ is Transitive

Hence $(P(S), \subseteq)$ is a Poset.

3.) The Set of all Positive Integers $\mathbb{Z}^+$ with divisibility $/$ relation i.e. $(\mathbb{Z}^+, /)$ is a Poset.

Since, for $a, b, c \in \mathbb{Z}^+$

i.) a / a , $\forall a \in \mathbb{Z}^+$

Reflexive

ii.) a / b and $b / a \Rightarrow a = b$

Antisymmetric

iii.) a / b and $b / c \Rightarrow a / c$

Transitive

Thus $/$ is a Partial order Relation on $\mathbb{Z}^+$ and $(\mathbb{Z}^+, /)$ is a Poset.

Note:- i.) $(\mathbb{Z}, /)$ is not Poset

as $a / (-a)$ and $(-a) / a$ but $a \neq -a$

So the relation is not antisymmetric on $\mathbb{Z}$

ii.) $(\mathbb{Z}^+, <)$ is not Poset

as the Relation is not Reflexive on $\mathbb{Z}^+$ for $a \not\prec a$ for any $a \in \mathbb{Z}^+$.

⑧ Comparability: —

The elements a and b of a Poset $(S, \leq)$ are comparable if either $a \leq b$ or $b \leq a$.

If $(S, \leq)$ is a Poset and every two elements of S are comparable. Then S is called a Totally ordered Set. It is also called a chain.

e.g. $\rightarrow (\mathbb{Z}, \leq)$ is a Totally ordered Set.

but $(\mathbb{Z}^+, /)$ is not Totally ordered set.

Product order $\rightarrow$ For $s, s' \in S$ and $t, t' \in T$

the Product Set defined by
 $(s, t) \leq (s', t')$ if $s \leq s'$ and $t \leq t'$
in S and T resp., where (s, t) and (s', t')
 $\in S \times T$. is called the Product order.

Lexicographic order $\rightarrow$

suppose $(S, \leq_1)$ and $(T, \leq_2)$
are Posets. The Cartesian Product $S \times T$
defined by $(s, t) \leq (s', t')$ either if $s \leq_1 s'$
or if both $s = s'$ and $t \leq_2 t'$.

⑨ Immediate Successor and Predecessor: —

Let $(X, <)$ be a Poset and suppose $x, y \in X$.
Then y is said to be immediate successor of x
if $x < y$ and there is no element z in X
such that $x < z < y$. we also say y covers
 x or x is an immediate Predecessor of y .

e.g. $\rightarrow$

In Poset $(P, \leq)$, $P = \{1, 2, 3, 4, 5\}$
the two elements $A = \{2, 4\}$ & $B = \{2, 4, 5\}$
are s.t. A is immediate Predecessor of B

(#) Hasse Diagram : $\rightarrow$

A Graphical representation of a Partial order Relation in which all arrow heads are understood to be pointing upward is known as the Hasse Diagram.

Procedure for Drawing Hasse Diagram : -

Step-1 Draw Digraph of the Relation

Step-2 Remove self loops

Step-3 Remove all Transitive edges

Step-4 Arrange all edges pointing upward and remove arrows from edges.

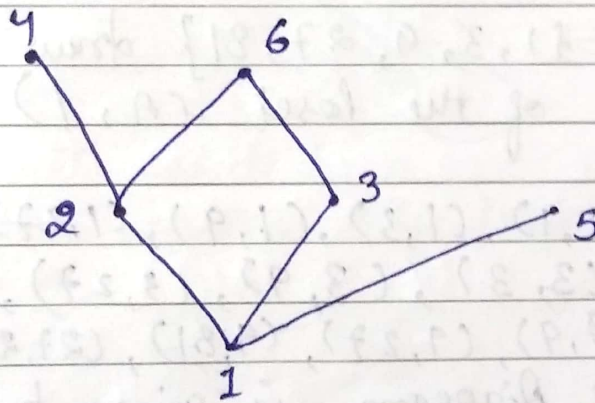
Step-5 Replace circle by dots or vertex.

Notes - Let $\leq$ be a Partial Relation on a Set X .
if $x, y \in X$ and y is an immediate successor of x , then y is placed at a higher level than x and x and y are joined by a straight line.

Ex-1. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $/$ is a partial order Relation on X . Draw the Hasse Diagram of $(X, /)$.

Soln:- $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

Here 2, 3, 5 are immediate successors of 1 and hence they are placed at higher level than 1 and connected with 1. Similarly, 6 is an immediate successor of 2, 3, 4 is an immediate successor of 2.



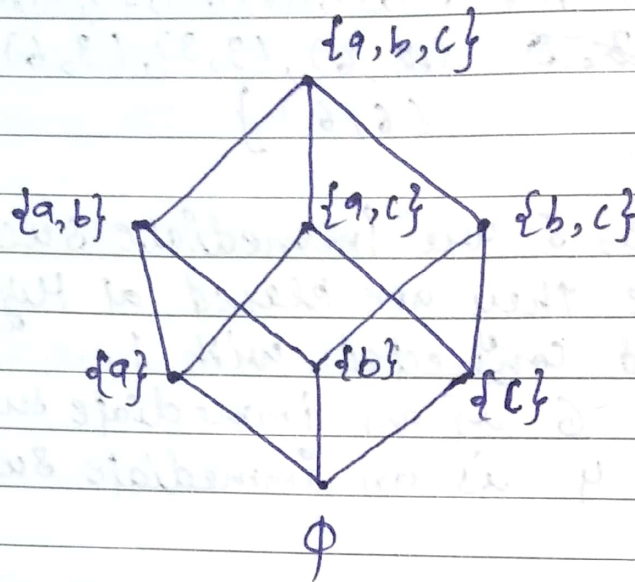
Ex-2. Draw the Hasse Diagram for the poset $(P(S), \subseteq)$, where $S = \{a, b, c\}$.

Soln:-

Here $P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$

Since $\emptyset$ is the subset of all sets, so it is the lowest point of the Diagram. Now $\{a\}, \{b\}, \{c\}$ are immediate successor of $\emptyset$. They are placed at higher level than $\emptyset$ and connected with $\emptyset$. $\{a, b\}$ is an immediate successor of $\{a\}$ and $\{b\}$ so $\{a, b\}$ is placed at higher level

than $\{a\}$ and $\{b\}$ and connected with $\{a\}, \{b\}$.
Similarly, the other points are drawn.
So the Hasse Diagram of the poset $(P(S), \subseteq)$ is

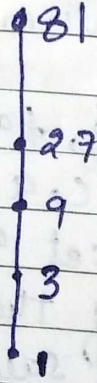


Ex:- 3. Let $A = \{1, 3, 9, 27, 81\}$ draw the Hasse Diagram of the poset $(A, |)$

Sol:-

$$R = \{(1, 1), (1, 3), (1, 9), (1, 27), (1, 81), (3, 3), (3, 9), (3, 27), (3, 81), (9, 9), (9, 27), (9, 81), (27, 27), (27, 81), (81, 81)\}$$

The Hasse Diagram is given by



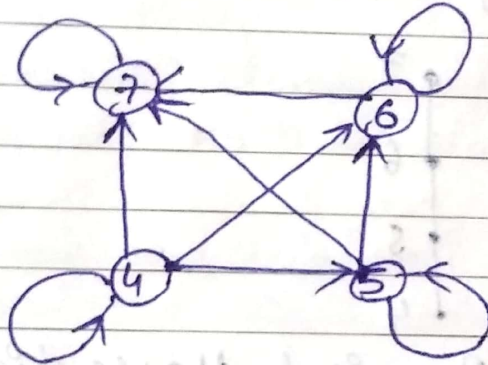
Ex:- 4. Consider the Set $A = \{4, 5, 6, 7\}$. Let R be the Relation $\leq$ on A . Draw the Hasse Diagram of R .

Ans -

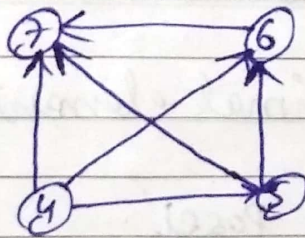
Given $A = \{4, 5, 6, 7\}$, $R \rightarrow \leq$
Then

$$R = \{(4, 4), (5, 5), (6, 6), (7, 7), (4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (6, 7)\}$$

Step 1 $\rightarrow$ Directed graph of $R \rightarrow$



Step 2:- Remove self loops.



Step 3:- Remove transitive edges.

The edges given by the Transitive Property

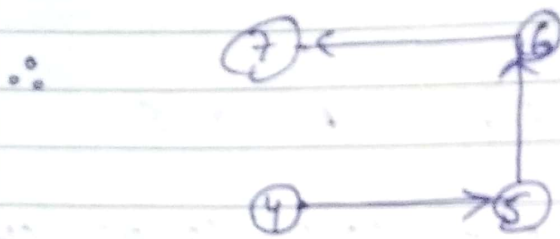
$$\{(4, 5), (4, 6), (5, 6), (5, 7), (6, 7), (4, 7)\}$$

$$(4, 5) \in R \text{ \& } (5, 6) \in R \Rightarrow (4, 6) \in R$$

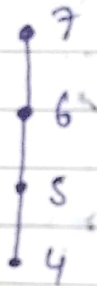
$$(5, 6) \in R \text{ \& } (6, 7) \in R \Rightarrow (5, 7) \in R$$

$$(4, 5) \in R \text{ \& } (5, 7) \in R \Rightarrow (4, 7) \in R$$

The ordered pairs implied by Transitivity
 $(4, 6), (4, 7), (5, 7)$



Step 4: - Remove arrows from edges and replace circle by dot and making all edges pointing upward.



This is the Required Hasse diagram of $(A, \leq)$

Special Elements in Posets: $\rightarrow$

⑧ maximal and minimal elements: -

Let $(P, \leq)$ be a poset.

An element a in the poset is called a maximal element of P if $a < x$ for no x in P , i.e. no element of P strictly succeeds a .

An element b in P is called minimal element of P if $x < b$ for no x in P , i.e. no element of P strictly precedes b .

⑧ Greatest and least element : —

- * An element $a \in P$ is Greatest element of P if $x \leq a$ for all $x \in P$ i.e every element in P precedes a .
- * An element $b \in P$ is called the Least element of P if $b \leq x$ for all $x \in P$ i.e every element in P succeeds b .

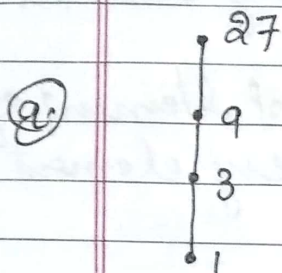
Note:- ① The Greatest and least elements are unique if they exist.

- ② The Greatest element if it exists, will be comparable with all elements of the poset.
- ③ Maximal and minimal elements are easy to spot using a Hasse Diagram. They are just the top and bottom elements in the diagram.
- ④ A poset may have more than one maximal and minimal elements.

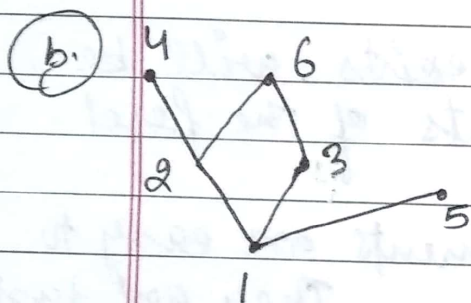
Ex:- Find the least and greatest element in the poset $(\mathbb{Z}^+, |)$ if they exist.

Sol:- Since $1/n$ for all $n \in \mathbb{Z}^+$
So 1 is the least element of $(\mathbb{Z}^+, |)$
Since there is no integer which is divisible by all positive integers
So there is no greatest element.

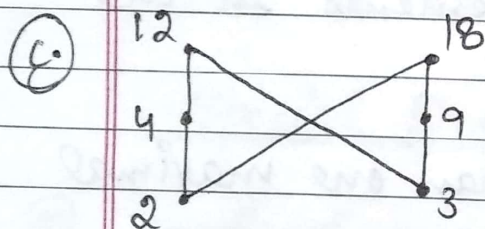
Exo:- Determine the greatest element, least element, maximal element and minimal elements for the Posets given by following Hasse Diagram.



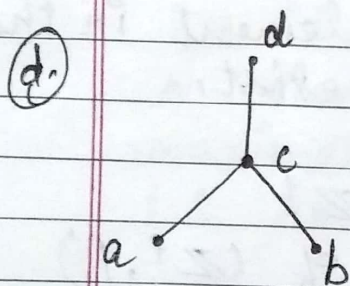
The least element of the Poset is 1 and is the only minimal element. The greatest element of the Poset is 27 and is the only Maximal element.



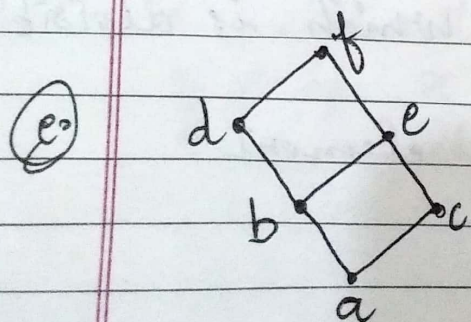
The least element is 1 and is the only minimal element. There is no greatest element and 4, 6, and 5 are maximal elements.



The Poset given by Hasse Diagram neither has least element nor has greatest element.
minimal elements = 2, 3
maximal elements = 12, 18



There is no least element and a and b are minimal elements. d is the greatest element and is the only maximal element.



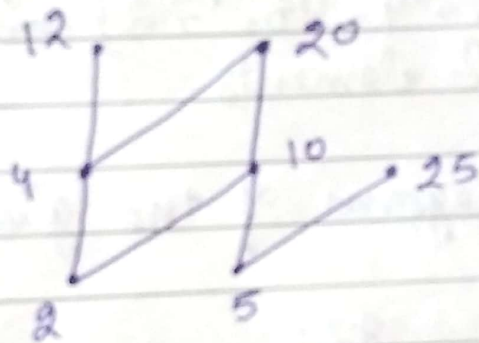
The greatest element is f and is the only maximal element. The least element is a and is the only minimal element.

The greatest element

Ques. 1 Find the maximal and minimal elements of the poset $(P, \leq)$ where $P = \{2, 4, 5, 10, 12, 20, 25\}$, and the partial order $\leq$ is 'a divides b'.

Soln. -

The Hasse diagram for the given poset $(P, \leq)$ is as follows:



maximal elements = 12, 20, 25
minimal elements = 2, 5

Ques. 2 In each of the following cases, give an example of a poset $(P, \leq)$ such that

- P has precisely three maximal elements and 2 minimal elements.
- P has minimal elements, but no maximal element.
- P has neither a minimal nor maximal element.

Soln. -

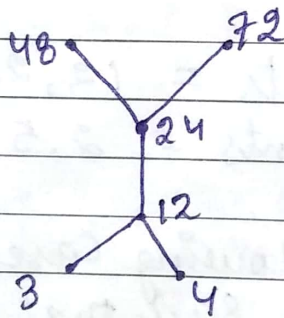
i.) The poset $P = \{2, 4, 5, 10, 12, 20, 25\}$ with relation divisibility ' $\mid$ ' is poset which has 3 maximal elements = 12, 20, 25 and 2 minimal elements = 2, 5

- ii) The poset $(\mathbb{Z}^+, \leq)$ has a minimal element 1 but no maximal elements.
whereas The poset $(\mathbb{Z}^-, \leq)$ has a maximal element -1 and no minimal elements.
- iii) The poset $(\mathbb{Z}, \geq)$ neither has a maximal element nor minimal element.

Ques 3 Draw the Hasse Diagram of the poset $A = \{3, 4, 12, 24, 48, 72\}$ and find maximal and minimal elements.

Solⁿ:

The Hasse Diagram of the given poset is as follows



maximal elements = 48, 72
minimal elements = 3, 4

There does not exist greatest and least elements

Upper and lower bound :-

Let B be a subset of a poset $(A, \leq)$
an element $u \in A$ is called an upper bound
of B if u succeeds every elements of B
i.e. $x \leq u$ for all $x \in B$.

An element $l \in A$ is called lower bound
of B if l precedes every element of B
i.e. $l \leq x$ for all $x \in B$.

Supremum = least upper bound (l.u.b.)
Infimum = greatest lower bound (g.l.b.)

Note :- (1) The Greatest element is always the
Supremum but the converse is not true.
 $a = \sup(B)$ is greatest element iff $a \in B$

(2) least element is always the Infimum
but the converse is not true.
 $b = \inf(B)$ is the least element iff $b \in B$.

Exo - Suppose the set $P = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is partially
ordered by the Relation "divides". Find the lower
and upper bounds of $A = \{1, 2\}$ and $B = \{3, 4, 5\}$
of the poset $(P, \leq)$. Also find the $\sup(A)$, $\inf(A)$,
 $\sup(B)$ and $\inf(B)$.

Soln :- upper bounds of $A = \{u \in P : x \leq u, \forall x \in A\}$
 $= \{2, 4, 6, 8\}$

so $\sup(A) = 2$
lower bounds of $A = \{l \in P : l \leq x, \forall x \in A\}$
 $= \{1\}$

so $\inf(A) = 1$

upper bounds of $B = \{u \in P : x \leq u, \forall x \in B\}$
 $= \emptyset$

So $\sup(B)$ does not exist

lower bounds of $A = \{l \in P : l \leq x, \forall x \in B\}$
 $= \{1\}$

So $\inf(B) = 1$

Ex:-2. Consider the poset $A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

1) find the greatest lower bound and the least upper bound of $A = \{6, 18\}$ and $B = \{4, 6, 9\}$.

Soln:-

lower bounds of $A = \{1, 2, 3, 6\}$

So $\inf(A) = 6$

upper bounds of $A = \{18, 36\}$

So $\sup(A) = 18$

Now lower bounds of $B = \{1\}$

So $\inf(B) = 1$

upper bounds of $B = \{36\}$

So $\sup(B) = 36$.

#

well-ordered Set $\rightarrow$

A set with an ordering relation is well-ordered if every non-empty subset of the set has a least element.

e.g $\rightarrow$ The set of natural numbers is well ordered.

But The set of Integers $\mathbb{Z}$ is not well ordered since the set of Negative Integers, which is a subset of $\mathbb{Z}$, has no least element.

#

Lattice $\rightarrow$ A poset $(P, \leq)$ is called a lattice if every 2-element subset of P has both a least upper bound and a greatest lower bound i.e. if $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ exist for every x and y in P . It is denoted by $(L, \leq, \vee, \wedge)$ where

$$x \vee y = \text{lub}\{x, y\} \quad (\text{read as } x \text{ join } y)$$

$$x \wedge y = \text{glb}\{x, y\} \quad (\text{read as } x \text{ meet } y).$$

Examples $\rightarrow$ ① The Power set $(P(S), \subseteq)$ is a lattice.

Let $A, B \subseteq S$ Then

$$\text{Sup}\{A, B\} = A \cup B = A \vee B$$

$$\text{Inf}\{A, B\} = A \cap B = A \wedge B$$

As, $A \subseteq A \cup B$ and $B \subseteq A \cup B$

and if $A \subseteq C$ and $B \subseteq C$ Then $A \cup B \subseteq C$.

So $A \cup B$ is the least subset of S which contains both A & B .

Also $A \cap B \subseteq A$ and $A \cap B \subseteq B$

and if $C \subseteq A$ & $C \subseteq B$ Then

$C \subseteq A \cap B$. So $A \cap B$ is the greatest subset of S which contained both A & B .

e.g In the poset $(P(\{a, b, c\}), \subseteq)$

$$\{a\} \vee \{b\} = \{a, b\}$$

$$\{a, b\} \vee \{b, c\} = \{a, b, c\}$$

$$\{a, b\} \wedge \{b, c\} = \{b\}$$

$$\{a\} \wedge \{b\} = \emptyset$$

Therefore $(P(S), \subseteq)$ is a lattice.

② The poset $(\mathbb{Z}^+, |)$ is a lattice because any upper bound of $\{a, b\}$ where $a, b \in \mathbb{Z}^+$ is nothing but an element which is divisible by both a and b i.e. common multiple of a and b which is an upper bound of a and b . The least upper bound is thus the least common multiple (lcm).
 Similarly, the greatest lower bound is the greatest common divisor (gcd) of a and b .

$$\begin{aligned} \therefore \text{lub } \{a, b\} &= a \vee b = \text{lcm}(a, b) \\ \text{glb } \{a, b\} &= a \wedge b = \text{gcd}(a, b) \end{aligned}$$

e.g $\rightarrow$ $\text{lub } \{6, 4\} = 6 \vee 4 = \text{lcm}(6, 4) = 12$
 $\text{glb } \{6, 4\} = 6 \wedge 4 = \text{gcd}(6, 4) = 2$

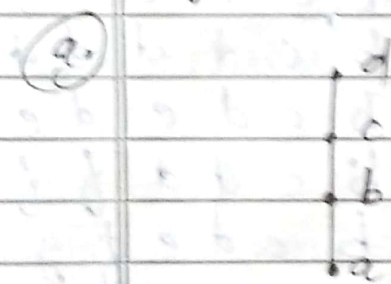
③ $(\mathbb{Z}, \leq)$ is a lattice where
 $m \vee n = \max\{m, n\}$
 $m \wedge n = \min\{m, n\}$

- (4) A chain $(P, \leq)$ is a lattice where $a \vee b = b$ and $a \wedge b = a$ if $a \leq b$ and $a \vee b = a$ and $a \wedge b = b$ if $b \leq a$.

Notes - Not all posets are lattice.

eg → The poset $(\{2, 3, 6, 12, 24, 36\}, /)$ is not lattice since $\text{glb}(2, 3)$ and $\text{lub}(24, 36)$ do not exist in S , hence $(S, /)$ is not a lattice.

Ques. Determine whether the posets represented by each of the Hasse Diagram are lattices.



Solⁿ - The closure table for $\text{lub}(v)$ and $\text{glb}(v)$ are given below

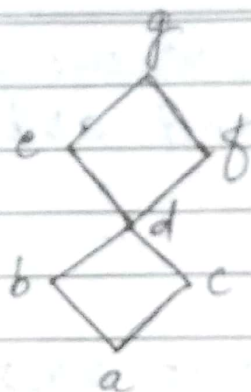
v	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

v	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

Since each subset of two elements has least upper bound and a greatest lower bound.
So The given Hasse Diagram Represents a lattice.

Date ____/____/____

(b)

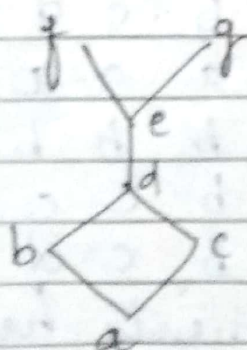


The closure table for $\text{lub}(v)$ and $\text{glb}(v)$ are as follows:

v	a	b	c	d	e	f	g
$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	e	e
f	a	b	c	d	e	f	f
g	a	b	c	d	e	f	g

Since each subset of two elements has least upper bound & a greatest lower bound. So this is a lattice.

(c)



$\text{lub}(v)$ table:

v	a	b	c	d	e	f	g
$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	e	e
f	a	b	c	d	e	f	f
g	a	b	c	d	e	f	g

g.l.b ($\wedge$) table:

$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	b	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	e	e
f	a	b	c	d	e	f	e
g	a	b	c	d	e	e	g

Since $\text{lub}(f, g)$ does not exist in poset
hence this is not a lattice.

Properties of Lattices: $\rightarrow$

Theorem 1: Let L be a lattice, then for every a and b in L .

- (a) $a \vee b = b$ if and only if $a \leq b$
- (b) $a \wedge b = a$ if and only if $a \leq b$
- (c) $a \wedge b = a$ if and only if $a \vee b = b$.

Proof: - (a) Let $a \vee b = b$
 $\Rightarrow$ Since $a \leq a \vee b = b$
 $\Rightarrow a \leq b$.

Conversely, let $a \leq b$

Since $a \leq b$ and $b \leq b$

$\Rightarrow b$ is an upper bound of a and b
 by def. of lub , $a \vee b \leq b$ — (1)

Also, $a \vee b$ is an upper bound
 $\Rightarrow b \leq a \vee b$ — (2)

So, from (1) and (2), we have

$$\boxed{a \vee b = b}$$
Hence proved.

(b) Let $a \wedge b = a$
 Since $a \wedge b$ is a lower bound of a and b
 $\therefore a \wedge b \leq b$

$$\Rightarrow a \leq b$$

conversely, if $a \leq b$ then since
 $a \leq a$ and $a \leq b$

$\Rightarrow a$ is a lower bound of a and b .

by def. of glb, $a \leq a \wedge b$ — (1)

also, $a \wedge b$ is a lower bound of a and b
 $\therefore a \wedge b \leq a$ — (2)

So, from (1) and (2), we have

$$\boxed{a \wedge b = a}$$

(c) From (a) & (b) we have
 $a \wedge b = a$ iff $a \leq b$ and $a \leq b$ iff $a \vee b = b$
 $\Rightarrow a \wedge b = a$ iff $a \vee b = b$.

Theorem 2: if L be any Lattice, then for any
 $a, b, c \in L$,

1. (a) $a \vee a = a$ (b) $a \wedge a = a$ Idempotency
2. (a) $a \vee b = b \vee a$ (b) $a \wedge b = b \wedge a$ Commutativity
3. (a) $(a \vee b) \vee c = a \vee (b \vee c)$
 (b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ Associativity
4. (a) $a \vee (a \wedge b) = a$ (b) $a \wedge (a \vee b) = a$ Absorption

Proof:- 1. (a) $a \vee a = \sup \{a, a\} = \text{lub} \{a, a\}$
 $= \text{lub} \{a\} = a.$

(b) $a \wedge a = \inf \{a, a\} = \text{glb} \{a, a\} = \text{glb} \{a\} = a.$

2. (a) $a \vee b = \text{lub} \{a, b\} = \text{lub} \{b, a\} = b \vee a$

(b) $a \wedge b = \text{glb} \{a, b\} = \text{glb} \{b, a\} = b \wedge a.$

3. (a) we have $(a \vee b) \vee c = \text{lub} \{a \vee b, c\}.$

$\Rightarrow a \vee b \leq (a \vee b) \vee c$ & $c \leq (a \vee b) \vee c$

also $a \leq (a \vee b)$ & $b \leq (a \vee b)$

So by transitivity

$a \leq (a \vee b) \vee c, b \leq (a \vee b) \vee c$

thus $(a \vee b) \vee c$ is an upper bound of a and b .

also $b \leq (a \vee b) \vee c$ & $c \leq (a \vee b) \vee c$

$\Rightarrow (a \vee b) \vee c$ is upper bound of b and c

by def. of lub of b & c i.e. $b \vee c,$

$b \vee c \leq (a \vee b) \vee c \Rightarrow (a \vee b) \vee c$ is an upper bound of $a, b \vee c$

& $a \leq (a \vee b) \vee c$

$\Rightarrow a \vee (b \vee c) \leq (a \vee b) \vee c$ — by def. of $\vee$ — (1)

Again, $a \vee (b \vee c) = \text{lub} \{a, b \vee c\}.$

$\Rightarrow a \leq a \vee (b \vee c)$ & $b \vee c \leq a \vee (b \vee c)$

also $b \leq b \vee c$ & $c \leq b \vee c$

$\Rightarrow b \leq a \vee (b \vee c)$ & $c \leq a \vee (b \vee c)$ by transitivity

$\Rightarrow a \vee (b \vee c)$ is an upper bound of b & c .

also $a \leq a \vee (b \vee c)$ & $b \leq a \vee (b \vee c)$

$\Rightarrow a \vee (b \vee c)$ is an upper bound of a & b .

by def. of $a \vee b, a \vee b \leq a \vee (b \vee c)$
 & $c \leq a \vee (b \vee c)$

$\Rightarrow a \vee (b \vee c)$ is an upper bound of $a \vee b$ and c

$$\Rightarrow (a \vee b) \vee c \leq a \vee (b \vee c) \quad \text{--- (2) by def of lub}$$

Applying Antisymmetry on (1) & (2), we have,

$$(a \vee b) \vee c = a \vee (b \vee c). \quad \text{Hence proved}$$

Theorem 3:- Let $(L, \leq)$ be a lattice, then for any $a, b, c, d \in L$.

1. $a \leq b \Rightarrow a \vee c \leq b \vee c$
2. $a \leq b \Rightarrow a \wedge c \leq b \wedge c$
3. $a \leq b$ and $c \leq d \Rightarrow a \vee c \leq b \vee d$.
4. $a \leq b$ and $c \leq d \Rightarrow a \wedge c \leq b \wedge d$.

Proof:-

1. we know,

$$a \leq b \Leftrightarrow a \vee b = b \quad \text{--- (1) (from Thm 1)}$$

Now, we get

$$\begin{aligned} (a \vee c) \vee (b \vee c) &= (a \vee c) \vee (c \vee b) \quad (\text{commutative law}) \\ &= a \vee (c \vee c) \vee b \quad (\text{Associative law}) \\ &= a \vee (c \vee b) \quad (\because c \vee c = c) \\ &= a \vee (b \vee c) \quad (\text{commutative law}) \\ &= (a \vee b) \vee c \quad (\text{Associative law}) \\ &= b \vee c \quad \text{from (1)} \end{aligned}$$

thus, $a \vee c \leq b \vee c$

2. we know,

$$a \leq b \Leftrightarrow a \wedge b = a \quad \text{--- (2) (from Thm 1)}$$

Now let $a \leq b$, we have to prove that

$$a \wedge c \leq b \wedge c$$

So consider, $(a \wedge c) \wedge (b \wedge c)$

$$\begin{aligned} &= (a \wedge c) \wedge (c \wedge b) \quad (\text{commutative law}) \\ &= (a \wedge (c \wedge c)) \wedge b \quad (\text{Associative law}) \\ &= a \wedge (c \wedge b) \quad (\because c \wedge c = c) \\ &= a \wedge (b \wedge c) \quad (\text{commutative law}) \\ &= (a \wedge b) \wedge c \quad (\text{Associative law}) \\ &= a \wedge c \quad \text{from (2)} \end{aligned}$$

$$\Rightarrow a \wedge c \leq b \wedge c.$$

3. Let $a \leq b$ and $c \leq d$ — (1)

we know

$$\left. \begin{aligned} a \leq b &\Leftrightarrow a \vee b = b \\ &\& c \leq d \Leftrightarrow c \vee d = d \end{aligned} \right\} \begin{array}{l} \text{(from Thm 1)} \\ \text{--- (2)} \end{array}$$

Now

$$\begin{aligned} (a \vee c) \vee (b \vee d) &= a \vee (c \vee b) \vee d \quad (\text{Associative law}) \\ &= a \vee (b \vee c) \vee d \quad (\text{commutative law}) \\ &= (a \vee b) \vee (c \vee d) \quad (\text{Associative law}) \\ &= b \vee d \quad \text{from (1) \& (2)} \\ \Rightarrow a \vee c &\leq b \vee d \quad (\text{from Thm 1.}) \\ &\text{i.e. } a \leq b \Leftrightarrow a \vee b = b \end{aligned}$$

4. Let $a \leq b$ and $c \leq d$ — (1)

we know,

$$\left. \begin{aligned} a \leq b &\Leftrightarrow a \wedge b = a \\ c \leq d &\Leftrightarrow c \wedge d = c \end{aligned} \right\} \text{--- (2)}$$

we have to prove that $a \wedge c \leq b \wedge d$

So consider

$$\begin{aligned} (a \wedge c) \wedge (b \wedge d) &= a \wedge (c \wedge b) \wedge d \quad (\text{Associative law}) \\ &= a \wedge (b \wedge c) \wedge d \quad (\text{commutative law}) \\ &= (a \wedge b) \wedge (c \wedge d) \quad (\text{Associative law}) \\ &= a \wedge c \quad \text{from (1) \& (2)} \\ \Rightarrow a \wedge c &\leq b \wedge d \end{aligned}$$

Theorem 4:- Let $(L, \leq, \vee, \wedge)$ be a lattice and $a, b, c \in L$. Then The distributive Inequalities are as follows:

(1.) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

(2.) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Proof:- (1.) Since $a \wedge b$ is greatest lower bound of a & b

$$\Rightarrow a \wedge b \leq a \quad \text{and} \quad a \wedge b \leq b$$

also, $b \vee c$ is least upper bound of b & c

$$\Rightarrow b \leq b \vee c \quad \text{and} \quad c \leq b \vee c$$

$$\text{So } a \wedge b \leq b \quad \text{and} \quad b \leq b \vee c$$

$$\Rightarrow a \wedge b \leq b \vee c \quad (\text{by transitivity})$$

$$\& \quad a \wedge b \leq a$$

$$\Rightarrow a \wedge b \text{ is lower bound of } \{a, b \vee c\}$$

$$\Rightarrow a \wedge b \leq a \wedge (b \vee c) \quad \text{--- (1)}$$

$$\text{Again } a \wedge c \leq a \quad \& \quad a \wedge c \leq c$$

$$\text{also } c \leq b \vee c$$

$$\Rightarrow a \wedge c \leq b \vee c \quad \text{by transitivity}$$

$$\& \quad a \wedge c \leq a$$

$$\Rightarrow a \wedge c \text{ is lower bound of } \{a, b \vee c\}$$

$$\Rightarrow a \wedge c \leq a \wedge (b \vee c) \quad \text{--- (2)}$$

(1) and (2) shows that

$$a \wedge (b \vee c) \text{ is an upper bound of } \{a \wedge b, a \wedge c\}$$

$$\Rightarrow (a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c) \quad \text{by def. of lub(v)}$$

Hence

$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

$$(2.) \quad a \leq a \vee b \quad \& \quad b \leq a \vee b$$

$$\text{and } b \wedge c \leq b \quad \& \quad b \wedge c \leq c$$

$$\text{So } b \wedge c \leq b \quad \& \quad b \leq a \vee b$$

$$\Rightarrow b \wedge c \leq a \vee b \quad (\text{by transitivity})$$

$$\& \quad a \leq a \vee b$$

$$\Rightarrow a \vee b \text{ is upper bound of } \{a, b \wedge c\}$$

$$\Rightarrow a \vee (b \wedge c) \leq a \vee b \quad \text{--- (1) def. of v}$$

$$\text{Again, } a \leq a \vee c$$

$$b \wedge c \leq b \leq a \vee c \Rightarrow b \wedge c \leq a \vee c$$

- $\Rightarrow a \vee c$ is upper bound of $\{a, b \wedge c\}$
 $\Rightarrow a \vee (b \wedge c) \leq a \vee c$ — (2) by def. of $\vee$
 (1) & (2) shows that
 $a \vee (b \wedge c)$ is lower bound of $\{a \vee b, a \vee c\}$
 $\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

hence proved

Remark: - These inequalities become equalities in many lattices (including the power set lattice) where the operation of union and intersection are distributive over each other. But they do not hold in general.

(#) Principle of duality: $\rightarrow$

Principle of duality for lattices states that if we interchange $\vee$ with $\wedge$ and $\leq$ with $\geq$ in a true statement about lattices, we get another true statement.

(#) Lattice Homomorphism (Homomorphism): -

Let L and M be two lattices then a mapping $f: L \rightarrow M$ is called lattice homo.

if

$$f(x \vee y) = f(x) \vee f(y) \quad (\text{join homomorphism})$$

and

$$f(x \wedge y) = f(x) \wedge f(y) \quad (\text{meet homomorphism})$$

$\forall x, y \in L$

$\rightarrow$ order-Homomorphism: - $f: L \rightarrow M$ is called order-Homomorphism if $x \leq y \Rightarrow f(x) \leq f(y)$ i.e. it preserves the partial order, hold $\forall x, y \in L$

→ If f is Homomorphism from L to M then $f(L)$ is called Homomorphic Image of L .

→ Every Lattice Homomorphism is an order - Homomorphism, But Converse is not True.

Isomorphism: — A Homomorphism $f: L \rightarrow M$ is called an Isomorphism if it is bijective i.e. one-one and onto.

Automorphism: — An Isomorphism $f: L \rightarrow L$ is called an Automorphism.

Lattice as Algebraic Structure : →

A lattice is an algebraic structure $(L, \vee, \wedge)$ with two binary operations $\vee$ and $\wedge$ which possesses the idempotent, commutative, Associative and Absorption Properties.

Sublattice : →

A non-empty subset L' of a lattice L is called a sublattice of L if $a, b \in L' \Rightarrow a \vee b, a \wedge b \in L'$ i.e. the Algebraic Structure $(L', \vee, \wedge)$ is a sublattice of $(L, \vee, \wedge)$ iff L' is closed under both operation $\vee$ and $\wedge$.

→ a sublattice itself is a lattice sub

→ Every singleton of a lattice L is a lattice of L .

Example:-1 Let S be a set and $L = (P(S), \cup, \cap)$ then only those sub collections of $P(S)$ are sublattices of L which are closed under $\cap$ and $\cup$. If $S = \{a, b, c, d\}$, then $\{\emptyset, \{a\}, \{a, c\}, \{c\}\}$ would be sublattice but $\{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}$ would not form a sublattice as $\{a\} \cup \{c\} = \{a, c\}$ is not there.

Union of two sublattices may not be sublattice.

Counter Example:-

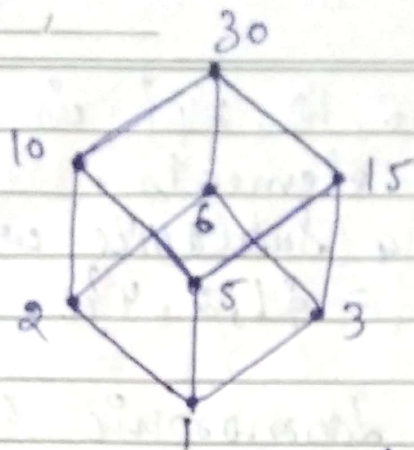
consider the lattice $L = \{1, 2, 3, 4, 6, 12\} = D_{12}$ of factors of 12 under divisibility. Then $S = \{1, 2\}$ and $T = \{1, 3\}$ are sublattices of L as $1 \cup 2 = \text{lcm}(1, 2) = 2 \in S$ & $1 \cap 2 = \text{gcd}(1, 2) = 1 \in S$ also $1 \cup 3 = \text{lcm}(1, 3) = 3 \in T$ & $1 \cap 3 = \text{gcd}(1, 3) = 1 \in T$.

But the union of S and T i.e. $S \cup T = \{1, 2, 3\}$ is not a sublattice as $2, 3 \in S \cup T$ but $2 \cup 3 = \text{lcm}(2, 3) = 6 \notin S \cup T$.

Let D_n denote the set of all positive divisors of n . Then D_m is a sublattice of D_n if $m|n$.

Ex:-

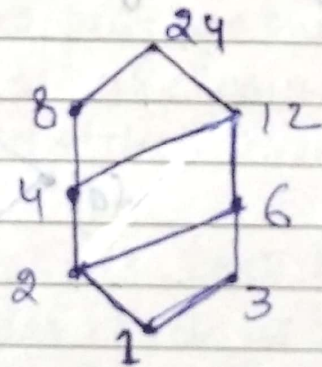
if $L = D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$, having the Hasse Diagram



then $D_6 = \{1, 2, 3, 6\}$, $D_{10} = \{1, 2, 5, 10\}$,
 $D_{15} = \{1, 3, 5, 15\}$ are sublattices of D_{30}
 $\{5, 10, 15, 30\}$ is also sublattice of D_{30} .

Ex:- Draw the Hasse Diagram of D_{24} and find
 its sublattices with 4, 5, 6 and 7 elements.
 Does D_{24} contain a sublattice with 3 elements.

Sol:- $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$



Hasse Diagram
 of D_{24} .

Since $6|24 \Rightarrow D_6$ is a sublattice of D_{24}
 So $D_6 = \{1, 2, 3, 6\}$ is a sublattice of D_{24}
 with 4 elements.

Since $12|24 \Rightarrow D_{12}$ is a sublattice of D_{24}
 So $D_{12} = \{1, 2, 3, 4, 6, 12\}$ is sublattice with
 6 elements

$S = \{1, 2, 3, 6, 12\}$ is a sublattice of D_{24}
 with 5 elements.

$T = \{1, 2, 3, 4, 6, 12, 24\}$ is a sublattice of D_4 with 7 elements.
also D_4 has a sublattice with 3 elements which is $D_2 = \{1, 2, 4\}$.

Example Based on Isomorphic lattice

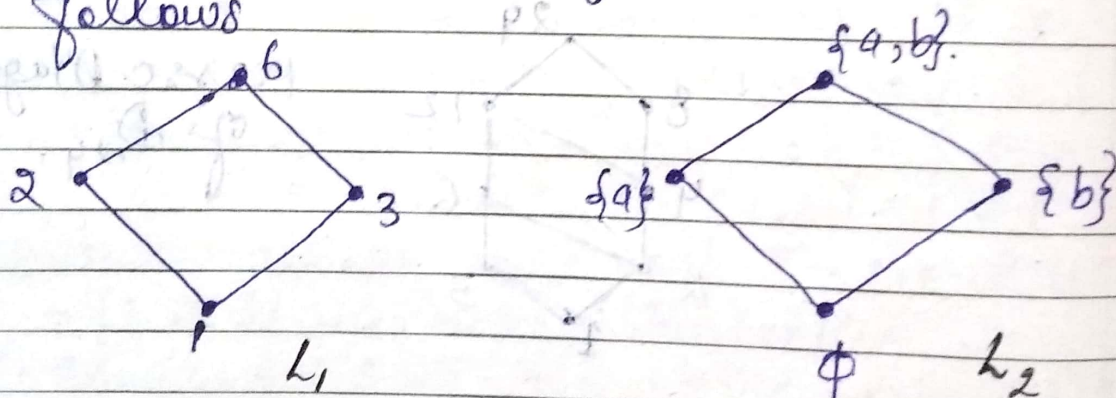
Ex:- Let L_1 be the lattice $D_6 = \{1, 2, 3, 6\}$ and L_2 be another lattice $(\mathcal{P}(S), \subseteq)$ where $S = \{a, b\}$. Then these two lattices are Isomorphic.

Soln:-

Here $L_1 = \{1, 2, 3, 6\}$

$L_2 = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

The Hasse Diagram of L_1 and L_2 are as follows



Let $f: L_1 \rightarrow L_2$ be defined by

$$f(1) = \phi, f(2) = \{a\}, f(3) = \{b\}, f(6) = \{a, b\}$$

Then f is one-one & onto

Now we have to show that

$$f(xy) = f(x) \cup f(y) \\ \& f(xy) = f(x) \cap f(y), \quad \forall x, y \in L_1$$

$$f(1 \vee 1) = f(1) = \phi \Rightarrow f(1 \vee 1) = f(1) \vee f(1)$$

$$f(1) \vee f(1) = \phi \vee \phi = \phi$$

$$f(1 \vee 2) = f(2) = \{a\} \Rightarrow f(1 \vee 2) = f(1) \vee f(2)$$

$$f(1) \vee f(2) = \phi \vee \{a\} = \{a\}$$

$$f(1 \vee 3) = f(3) = \{b\} \Rightarrow f(1 \vee 3) = f(1) \vee f(3)$$

$$f(1) \vee f(3) = \phi \vee \{b\} = \{b\}$$

$$f(1 \vee 6) = f(6) = \{a, b\} \Rightarrow f(1 \vee 6) = f(1) \vee f(6)$$

$$f(1) \vee f(6) = \phi \vee \{a, b\} = \{a, b\}$$

Similarly we can do for $\wedge$

Now,

$$f(2 \vee 3) = f(6) = \{a, b\} \Rightarrow f(2 \vee 3) = f(2) \vee f(3)$$

$$f(2) \vee f(3) = \{a\} \vee \{b\} = \{a, b\}$$

$$f(2 \wedge 3) = f(1) = \phi \Rightarrow f(2 \wedge 3) = f(2) \wedge f(3)$$

$$f(2) \wedge f(3) = \{a\} \wedge \{b\} = \phi$$

we can also prove for the pairs $(2, 6)$ & $(3, 6)$ under $\vee$ & $\wedge$.

Since the mapping is one-one & onto & preserves closure property under $\vee$ & $\wedge$.
Hence the lattices are isomorphic.

* Isomorphic lattices look identical except for the label of nodes. Each is a mirror image of the other with the elements simply relabelled.

* If S and T are any two finite sets with n elements. Then lattices $(P(S), \subseteq)$ and $(P(T), \subseteq)$ are isomorphic.

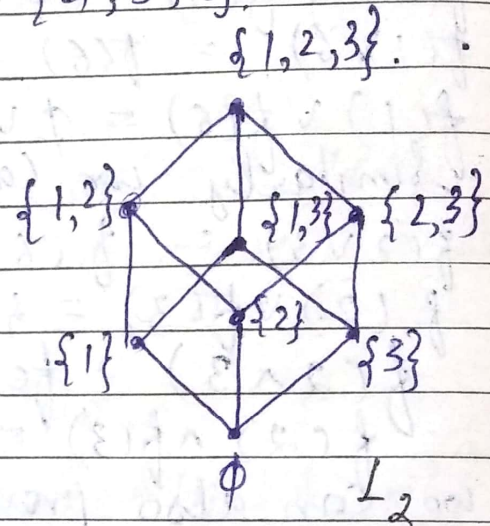
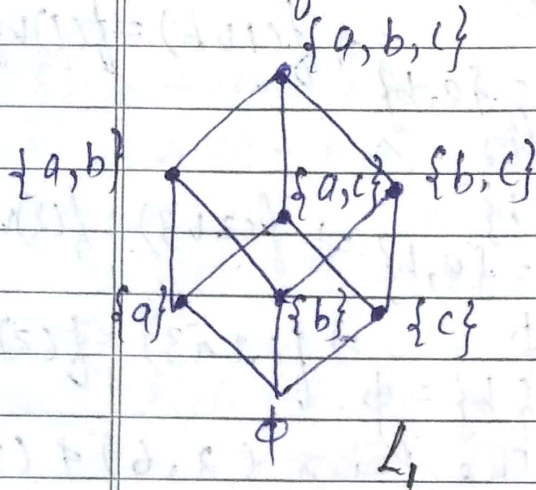
e.g. — if $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$.
Then

$$L_1 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$$

$$L_2 = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$$

Let $f: L_1 \rightarrow L_2$ is an isomorphism defined by

$$\begin{aligned} f(\{1\}) &= \{a\}, & f(\{2\}) &= \{b\}, & f(\{3\}) &= \{c\}, \\ f(\{1,2\}) &= \{a,b\}, & f(\{1,3\}) &= \{a,c\}, \\ f(\{2,3\}) &= \{b,c\}, \\ f(\{1,2,3\}) &= \{a,b,c\}. \end{aligned}$$



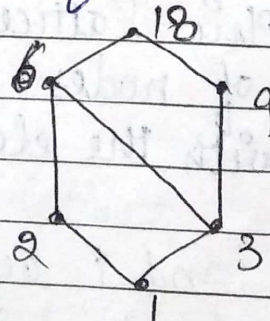
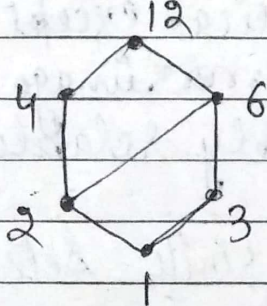
Ex:-

Show that D_{12} and D_{18} are isomorphic lattices.

Here $D_{12} = \{1, 2, 3, 4, 6, 12\}$

$D_{18} = \{1, 2, 3, 6, 9, 18\}$

The Hasse Diagram of D_{12} & D_{18} are as



we define a mapping $f: D_{12} \rightarrow D_{18}$ as

$$f(1) = 1, \quad f(2) = 3, \quad f(3) = 2,$$

$$f(4) = 9, \quad f(6) = 6, \quad f(12) = 18$$

which is one-one and onto mapping

Types of Lattices : $\rightarrow$

Complete Lattice : - A lattice is called complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.

Every finite lattice is complete
b/c Every subset here is finite.

Every complete lattice must have a least element and a greatest element.

The least and the greatest elements of lattice are called bounds (units, universal bounds) of a lattice and are denoted by 0 and 1 respectively.

Bounded Lattice : - A lattice which has both elements 0 and 1 is called a Bounded lattice and is often denoted by $(L, \vee, \wedge, 0, 1)$.

where $0 =$ least element of lattice
 $1 =$ greatest element of lattice.

- Every finite lattice is bounded.

Let $(L, \vee, \wedge)$ be a finite lattice
with $L_n = \{a_1, a_2, \dots, a_n\}$

Then

least element of $L = \bigwedge_{i=1}^n a_i = a_1 \wedge a_2 \wedge \dots \wedge a_n = 0$

and greatest element of L

$$= \bigvee_{i=1}^n a_i = a_1 \vee a_2 \vee \dots \vee a_n = 1$$

& both of them exist in L

So L is a Bounded Lattice.

Exo (1) $(P(S), \cup, \cap)$ is bounded lattice
with $\phi = 0$ & $S = 1$.

$$\text{Let } P(S) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\} \}$$

$$\text{where } S = \{a, b, c\}.$$

Here the greatest element $= S = \{a, b, c\} = 1$

& the least element $= \phi = 0$

(2.) $(\mathbb{Z}^+, |)$ is not bounded lattice.

$$\mathbb{N} = \mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$$

It has least element 1

but $\nexists$ any greatest element.

So $(\mathbb{Z}^+, |)$ is not bounded lattice.

(3.) $(\mathbb{Z}, \leq)$ is not bounded lattice.

Since It has neither greatest element
nor least element.

(*) In a Bounded lattice, the bounds 0 and 1
clearly satisfy following conditions:

$$0 \wedge a = 0 = a \wedge 0, \quad \forall a \in L$$

$$0 \vee a = a = a \vee 0, \quad \forall a \in L$$

$$1 \wedge a = a = a \wedge 1, \quad \forall a \in L$$

$$1 \vee a = 1 = a \vee 1, \quad \forall a \in L$$

Date: / /

Complement of an element:

In a bounded lattice $(L, \vee, \wedge, 0, 1)$, an element $b \in L$, is called a complement of an element $a \in L$ if

$$a \wedge b = 0 \quad \text{and} \quad a \vee b = 1.$$

where 0 and 1 are least and greatest elements.

Note:- (1) if b is complement of a then a is also complement of b .

(2) Any element $a \in L$ may or may not have a complement.

(3) An element $a \in L$ may have more than one complement in L .

(4) In any bounded lattice, the bound 0 & 1 are unique complements of each other b/c

$$0 \vee 1 = 1 \quad \& \quad 0 \wedge 1 = 0$$

Complemented lattice: $\rightarrow$

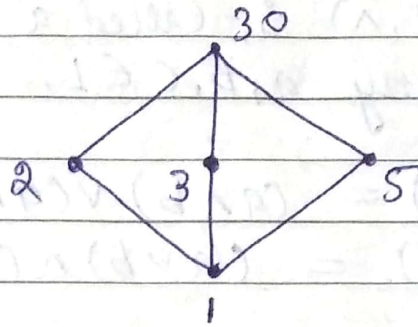
A lattice $(L, \vee, \wedge)$ is called complemented lattice if

(i) L is bounded.

(ii) every element $a \in L$ has a complement

Ex:- Let $P = \{1, 2, 3, 5, 30\}$ with Relation R defined by 'a divides b' be a poset. Then $(P, |)$ is bounded and complemented lattice.

Solⁿ: Given that $P = \{1, 2, 3, 5, 30\}$.
 Then the Hasse Diagram of $(P, \mid)$ is as follows:



This poset is a lattice as for every pair (a, b) . The $\text{Sup}\{a, b\}$ & $\text{Inf}\{a, b\}$ exist in P .
 And this lattice is Bounded as it has greatest element = 30 and least element = 1.

Now, since $2 \vee 3 = 30$ & $2 \vee 5 = 30$
 & $2 \wedge 3 = 1$ & $2 \wedge 5 = 1$

$\Rightarrow$ 2 has two complements 3 & 5.

& 3 has also two complements 2 & 5.

5 has two complements 2 & 3.

1 has one complement 30.

30 has one complement 1.

As $1 \vee 30 = 30$ & $1 \wedge 30 = 1$.

Hence this lattice is complemented.

distributive lattice $\rightarrow$

A Lattice $(L, \vee, \wedge)$ is called a distributive lattice if for any $a, b, c \in L$.

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Note:-

Since both the inequalities are equivalent to one another, hence to check whether the lattice is distributed or not, it is sufficient to verify one of them.

Ex:- 1. $L = (P(S), \vee, \wedge)$ is distributive lattice

$P(S)$ is the set of all subsets of S

Let A, B and $C \in P(S)$. Then

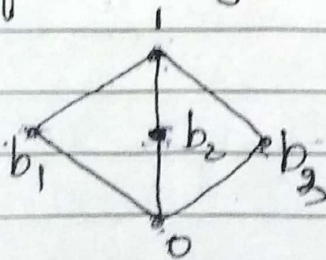
$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

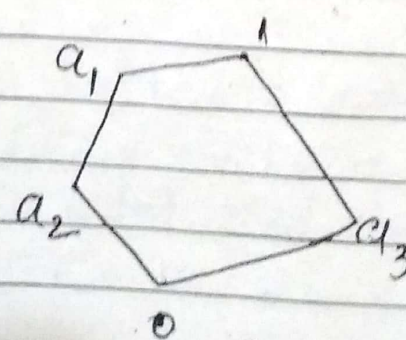
for elements A, B, C in $P(S)$.

$\Rightarrow$ L is distributive lattice.

Ex:- 2. check whether the lattices given in the following diagrams are distributive.



(a) Diamond



(b) Pentagon

Soln (a) $b_1 \wedge (b_2 \vee b_3) = b_1 \wedge 1 = b_1$ ($\because b_2 \vee b_3 = 1$)
 $= b_1$

but $(b_1 \wedge b_2) \vee (b_1 \wedge b_3) = 0 \vee 0 = 0$

$\Rightarrow b_1 \wedge (b_2 \vee b_3) \neq (b_1 \wedge b_2) \vee (b_1 \wedge b_3)$

Hence this lattice is not Distributive.

(b) $a_1 \wedge (a_2 \vee a_3) = a_1 \wedge 1 = a_1$ ($\because a_2 \vee a_3 = 1$)
 but $(a_1 \wedge a_2) \vee (a_1 \wedge a_3) = a_2 \vee 0 = a_2$ ($\because a_1 \wedge a_2 = a_2$
 $\& a_1 \wedge a_3 = 0$)

$\therefore a_1 \wedge (a_2 \vee a_3) \neq (a_1 \wedge a_2) \vee (a_1 \wedge a_3)$

Hence the given lattice (pentagon) is not Distributive.

Theorem:- A Lattice $(L, \vee, \wedge)$ is Distributive iff it does not contain the five element pentagonal, or, the Diamond Lattice as one of its sublattices. (or an isomorphic copy thereof)

Th^m:- In a Distributive lattice, if an element has a complement then this complement is unique.

or

A lattice L is called Distributive if every element of L has at most one ^{comp} complement.

Proof:- Suppose an element $a \in L$ has two complements b and c then

$a \vee b = 1$ and $a \wedge b = 0$

$\& a \vee c = 1$ and $a \wedge c = 0$

we have $b = b \wedge 1$

$= b \wedge (a \vee c)$

$= (b \wedge a) \vee (b \wedge c)$ (by distributive law)

$$= (a \wedge b) \vee (b \wedge c)$$

$$= 0 \vee (b \wedge c)$$

$$(\because a \wedge b = 0)$$

$$= (a \wedge c) \vee (b \wedge c)$$

$$= (a \vee b) \wedge c$$

$$= 1 \wedge c$$

$$= c$$

$$\Rightarrow b = c$$

Hence, In a distributive lattice, Every element has a unique complement if it exist.

Modular Lattice: $\rightarrow$

A Lattice $(L, \wedge, \vee)$ is said to be modular if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$ for all $a, b, c \in L$.

Ex:- The Lattice given by the following diagram is modular

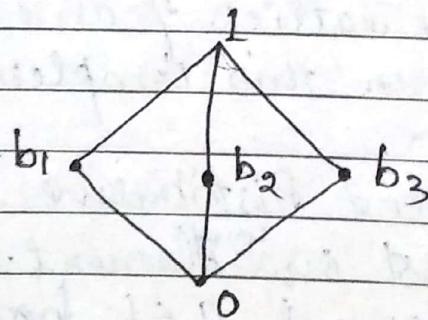


Diagram
Diamond
Lattice.

Sol:- This lattice is not Distributive.

Since the lattice is symmetric w.r.t b_1, b_2, b_3 .
So the only situations for $a \leq c$ (condition of modularity) are $b_1 \leq 1$ and $0 \leq b_1$.

$$(a \vee b) \wedge c = a \vee (b \wedge c)$$

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Thus taking $a = b$, and $c = 1$.

$$a \vee (b \wedge c) = b \vee (b \wedge 1) = b \vee b \quad (\because b \wedge 1 = b)$$

$$\text{and } (a \vee b) \wedge c = (b \vee b) \wedge 1 = b \vee b.$$

Whatever b may be.

Similarly for $a = 0$ and $c = b$,

$$a \vee (b \wedge c) = 0 \vee (b \wedge b) = b \wedge b,$$

$$(a \vee b) \wedge c = (0 \vee b) \wedge b = b \wedge b,$$

Whatever b may be.

$\Rightarrow$ modularity holds.

Hence The Diamond lattice is Modular.

Imp

Theorem:- Every Distributive lattice is modular.

Proof:-

Let $(L, \leq)$ be a distributive lattice and

$a, b, c \in L$ be such that $a \leq c$.

$$\text{Thus if } a \leq c \Rightarrow a \vee c = c \quad \text{--- (1)}$$

Now

$$\begin{aligned} a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \quad \because L \text{ is distributive} \\ &= (a \vee b) \wedge c \quad \text{from (1)} \end{aligned}$$

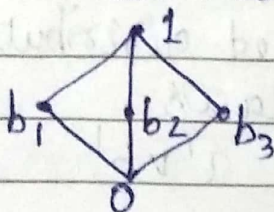
$$\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c \text{ whenever } a \leq c$$

Hence, every distributive lattice is modular.

Note:- The Converse of above Th^m is not True
i.e. A modular lattice may not be distributive

e.g. -

The modular The Diamond lattice



is modular but not Distributive.

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Ex: - Prove that a lattice $(L, \leq)$ is modular iff $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee (a \wedge c))$ $\forall a, b, c \in L$.

Sol: - Suppose $(L, \leq)$ is modular then
 $(a \wedge b) \vee (a \wedge c) = (a \wedge c) \vee (a \wedge b)$ comm. law
 $= (a \wedge c) \vee (b \wedge a)$
 $= ((a \wedge c) \vee b) \wedge a$ By modularity
 $\because a \wedge c \leq a$
 $= a \wedge (b \vee (a \wedge c))$ by commutative law.

Conversely, suppose that $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee (a \wedge c))$ for all $a, b, c \in L$.

Let $a, b, c \in L$ be such that $a \leq c$

Then $a \wedge c = a$

Now $(c \wedge b) \vee (c \wedge a) = (a \wedge c) \vee (b \wedge c)$
 $= c \wedge (b \vee (a \wedge c))$ by comm. law

Hence, $(c \wedge b) \vee a = c \wedge (b \vee a)$
 i.e. $a \vee (b \wedge c) = (a \vee b) \wedge c$

Ex: - The pentagonal lattice is not modular.

Th^m: - A lattice $(L, \wedge, \vee)$ is modular iff it does not have a sublattice isomorphic with the pentagonal lattice.

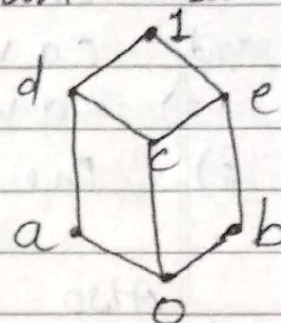
Th^m: - If $(L, \vee, \wedge)$ is a complemented distributive lattice, then De Morgan's laws $(a \vee b)' = a' \wedge b'$ & $(a \wedge b)' = a' \vee b'$ holds for all $a, b \in L$.

Ex:- Show that the poset given in the following Hasse Diagram is a lattice. It is distributive and complemented?

Solution:- Here 0 is the least element & 1 is the greatest element.

Now to check whether the given Hasse Diagram represents lattice or not.

We form the tables for $\text{lub}(v)$ and $\text{glb}(v)$ of pairs of elements of the set $S = \{0, a, b, c, d, e, 1\}$.



$\text{lub}(v)$	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	a	1	d	d	1	1
b	b	1	b	e	1	e	1
c	c	d	e	c	d	e	1
d	d	d	1	d	d	1	1
e	e	1	e	e	1	e	1
1	1	1	1	1	1	1	1

$\text{glb}(v)$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	0	0	a	0	a
b	0	0	b	0	0	b	b
c	0	0	0	c	c	c	c
d	0	a	0	c	d	e	d
e	0	0	b	c	c	e	e
1	0	a	b	c	d	e	1

Since lub and glb of every pair of elements, hence the poset is a lattice.

The lattice is not Distributive as
 $a \vee (b \wedge c) = a \vee 0 = a$
 and $(a \vee b) \wedge (a \vee c) = 1 \wedge d = d$
 $\Rightarrow a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$
 $\Rightarrow$ The lattice is not distributive.

Also The lattice is not complemented
 as c has no complement
 as $c \vee x \neq 1$ for any $x \neq 1, x \in S$

1	a	b	c	d	e	0	$\vee$
1	a	b	c	d	e	0	a
1	1	b	b	1	a	a	1
1	a	1	a	d	1	d	d
1	a	b	c	a	d	c	1
1	1	b	b	1	b	b	b
1	a	1	a	a	1	a	a
1	1	1	1	1	1	1	1

1	a	b	c	d	e	0	$\wedge$
a	a	a	a	a	a	a	a
a	a	a	a	a	a	a	a
d	d	a	a	d	a	a	d
a	a	a	a	a	a	a	a
b	a	b	a	a	a	a	b
a	a	a	a	d	a	a	a
1	a	b	c	d	e	0	1